

# Lifschitz singularity for subordinate Brownian motions in presence of the Poissonian potential on the Sierpiński gasket

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## Abstract

We establish the Lifschitz-type singularity around the bottom of the spectrum for the integrated density of states for a class of subordinate Brownian motions in presence of the nonnegative Poissonian random potentials, possibly of infinite range, on the Sierpiński gasket. We also study the long-time behaviour for the corresponding averaged Feynman-Kac functionals.

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## 1 Introduction

The integrated density of states is one of central objects in the physics of large-volume systems, especially systems with in-built randomness. The randomness can come from the interaction with external force field, described by its potential  $V$ . This leads us to the study of random Hamiltonians, in particular those of Schrödinger type: given a sufficiently regular, possibly random, potential  $V$  one considers the operator

$$H := H_0 + V,$$

where  $H_0$  is the Hamiltonian of the system with no potential interaction. The best analyzed situation is that of  $H_0 = -\Delta$  (in various state-spaces  $X$ ). The spectrum of  $H$  is typically not discrete. Moreover, spectral properties of such infinite-volume (i.e. defined with the whole space  $X$ ) Schrödinger operators are usually difficult to handle. The notion of the integrated density of states can come to the rescue: it captures some of the properties of the spectral distribution, while being easier to calculate and easier to work with [4, Chapter VI].

Informally speaking, one considers operators  $H$  restricted to a finite volume  $\Omega \subset X$ , build empirical measures  $l_\Omega$  based on the spectra of these operators normalized by the volume of  $\Omega$ , and then one takes the limit of  $l_\Omega$ , in appropriate sense, when  $\Omega \nearrow X$ . The resulting limit (if it exists) is called the

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integrated density of states (IDS, for short). Same procedure can be performed for random potentials  $V^\omega$  – in this case one is interested in the almost-sure limiting behavior of measures  $l_\Omega$ . When the potential  $V^\omega$  exhibits some ergodicity properties then the limit can be nonrandom.

This paper is concerned with random Schrödinger operators with nonnegative Poissonian potentials. In this case, the existence of the nonrandom IDS is a common feature and for  $H_0 = -\Delta$  has been proven e.g. in the Euclidean space [15], hyperbolic space [24], the Sierpiński gasket [19], other nested fractals [21]. In all these situations one has the so-called Lifschitz singularity: the rate of decay of the IDS at the bottom of the spectrum is faster than this of the IDS for the system without external interaction. Note also that the Lifschitz singularity is closely related to the behaviour of the so-called Wiener sausage when  $t \rightarrow \infty$  (for the sausage asymptotics in the classical case see [8]).

While the IDS based on the Laplacian is fairly well understood (see e.g. [4], [23]), it is not so for the IDS based on nonlocal operators. In the case of Lévy processes on  $\mathbb{R}^d$ , the existence and asymptotical properties of IDS with Poissonian potentials have been established in [16, 17]. Up to date, there were no results concerning the ‘nonlocal IDS’ on irregular sets, such as fractals. Recently, we have proven the existence of the IDS for subordinate Brownian motions on the Sierpiński gasket perturbed by Poissonian potentials with two-argument profiles  $W$  that may have infinite range and local singularities [11]. The Lifschitz tail for stable processes on the Sierpiński gasket evolving among killing Poissonian obstacles was derived in [14]. The present paper is meant as the continuation of [11] in the potential case: under appropriate assumptions on the potential  $V$  (expressed in terms of its profile function  $W$ ) and the Laplace exponent of the subordinator  $S$  (assumed to be a complete Bernstein function), we analyse the asymptotical behaviour of the IDS based on the generator of the resulting subordinate Brownian motion evolving in presence of the potential  $V$ . We first estimate the Laplace transform of the IDS (Theorems 3.1 and 4.3), and then use exponential Tauberian theorems from [9] to transform them into estimates on the IDS itself (Theorems 5.1 and 5.2). The proof of lower bound seems to be easier, while for the proof of upper bound we need to reduce the problem to study the subordinate Brownian motion reflected in a gasket triangle of size  $2^M$  perturbed by some special periodization of the initial potential (Lemma 4.4), and compare the principal eigenvalue of its generator with the principal eigenvalue related to the stable process reflected in the unit triangle, with rescaled potential (Lemmas 4.3 and 4.5). After this simplification, we can just proceed with stable process and employ the coarse-graining technique (‘the enlargement of obstacles method’) of Sznitman [25], adapted recently to the case of non-diffusive processes in [13]. In present work, Sznitman’s theorem is needed for the potential case (the original work [25] was concerned with killing obstacles). To make the paper self-contained, we give a proof of the desired theorem (Theorem 7.1) in the Appendix. We also derive the estimates for the corresponding averaged Feynman-Kac functional, which can be interpreted as the survival probability of the process killed by the potential  $V$  up to time  $t$  (Theorem 3.2 and Remark 4.1). Our proof of the upper bound hinges on properties of the reflected subordinate Brownian motions on the gasket. Construction of such processes relies on the specific geometry of the gasket and does not seem to have an obvious extension to more general fractals.

A remarkable feature of our results is that they take into account also the long range interaction which comes from the decay rate of the profile  $W$  at infinity and often has a decisive impact on the properties of the IDS. This seems to be new even in the case of Brownian motion on the Sierpiński gasket. In the jump case, our bounds reflect very well the competition between intensity of large jumps of the process and the rate of killing of the Poissonian potential given by the tail properties of the potential profile. Our approach covers a wide range of jump subordinators with drift (the resulting subordinate process is then often called *jump diffusion*), as well as purely jump subordinators including stable, mixture of stable, some logarithmic-stable and others (see the examples in Section 6). Unfortunately, our Theorem 4.3 cannot be applied to the relativistic stable subordinators. We believe that this result holds true in this case as well, but the proof would require tools specialized to those specific processes, not available yet.

## 2 Preliminaries

### 2.1 The Sierpiński gasket

The infinite Sierpiński Gasket we will be working with is defined as a blowup of the unit gasket, which in turn is defined as the fixed point of the hyperbolic iterated function system in  $\mathbb{R}^2$ , consisting of three maps:

$$\phi_1(x) = \frac{x}{2}, \quad \phi_2(x) = \frac{x}{2} + \left(\frac{1}{2}, 0\right) \quad \phi_3(x) = \frac{x}{2} + \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

The unit gasket,  $\mathcal{G}_0$ , is the unique compact subset of  $\mathbb{R}^2$  such that  $\mathcal{G}_0 = \phi_1(\mathcal{G}_0) \cup \phi_2(\mathcal{G}_0) \cup \phi_3(\mathcal{G}_0)$ . Then we set:

$$\mathcal{G}_n = 2^n \mathcal{G}_0 = ((\phi_1^{-1}))^n(\mathcal{G}_0), \quad \mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n.$$

All the triangles of size  $2^M$ ,  $M \in \mathbb{Z}$ , that build up the infinite gasket will be denoted by  $\mathcal{T}_M$ , and the collection of their vertices – by  $\mathcal{V}_M$ .

We equip the gasket with the shortest path distance  $d(\cdot, \cdot)$ : for  $x, y \in \bigcup_M \mathcal{V}_M$ ,  $d(x, y)$  is the infimum of Euclidean lengths of all paths, joining  $x$  and  $y$  on the gasket. For general  $x, y \in \mathcal{G}$ ,  $d(x, y)$  is obtained by a limit procedure. This metric is equivalent to the usual Euclidean metric inherited from the plane. Observe that  $\mathcal{G}_M = B(0, 2^M)$ , where the ball is taken in either the Euclidean or the shortest path metric.

By  $m$  we denote the Hausdorff measure on  $\mathcal{G}$  in dimension  $d = \frac{\log 3}{\log 2}$ , normalized to have  $m(\mathcal{G}_0) = 1$ . The number  $d$  is called the fractal dimension of  $\mathcal{G}$ . Another characteristic number of  $\mathcal{G}$ , namely  $d_w = \frac{\log 5}{\log 2}$  is called the walk dimension of  $\mathcal{G}$ . The spectral dimension of  $\mathcal{G}$  is  $d_s = \frac{2d}{d_w}$ . The measure  $m$  is a  $d$ -measure, i.e. there exists a constant  $c_{2.1} > 0$  such that for all  $x \in \mathcal{G}$ ,  $r > 0$  one has

$$c_{2.1}^{-1} r^d \leq m(B(x, r)) \leq c_{2.1} r^d \quad (2.1)$$

By an elementary slicing argument we obtain the following bound, valid for  $x \in \mathcal{G}$ ,  $\lambda > 0$ ,  $a > 0$ :

$$c_{2.2}^{-1} \frac{1}{a^\lambda} \leq \int_{d(x, y) > a} \frac{dm(y)}{d(x, y)^{d+\lambda}} \leq c_{2.2} \frac{1}{a^\lambda}, \quad (2.2)$$

where  $c_{2.2} = c_{2.2}(d, \lambda) > 0$  is a numerical constant.

In the sequel, we will need a projection from  $\mathcal{G}$  onto  $\mathcal{G}_M$ ,  $M = 0, 1, 2, \dots$ . To define it properly, we first put labels on the set  $\mathcal{V}_0$  (see [19]).

Observe that  $\mathcal{G}_0 \subset (\mathbb{Z}_+)e_1 + (\mathbb{Z}_+)e_2$ , where  $e_1 = (1, 0)$  and  $e_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ . Next, consider the commutative 3-group  $\mathbb{A}_3$  of even permutations of 3 elements,  $\{a, b, c\}$ , i.e.  $\mathbb{A}_3 = \{id, (a, b, c), (a, c, b)\}$ , and we denote  $p_1 = (a, b, c)$ ,  $p_2 = (a, c, b)$ . The mapping

$$\mathcal{V}_0 \ni x = ne_1 + me_2 \mapsto p_1^n \circ p_2^m \in \mathbb{A}_3$$

is well defined, and for  $x \in \mathcal{V}_0$  we put  $l(x) = (p_1^n \circ p_2^m)(a)$ . This way, every triangle of size 1 has its vertices labeled ‘ $a, b, c$ ’, see [19, Fig. 4 and 5]. Note that this property extends to every triangle of size  $2^M$ , which corresponds to putting labels on the elements of  $\mathcal{V}_M$ : every triangle of size  $2^M$  has three distinct labels on its vertices.

Let  $M \geq 0$  be fixed. For  $x \in \mathcal{G} \setminus \mathcal{V}_M$ , there is a unique triangle of size  $2^M$  that contains  $x$ ,  $\Delta_M(x)$ , and so  $x$  can be written as  $x = x_a a(x) + x_b b(x) + x_c c(x)$ , where  $a(x)$ ,  $b(x)$ ,  $c(x)$  are the vertices of  $\Delta_M(x)$  with labels  $a, b, c$  and  $x_a, x_b, x_c \in (0, 1)$ ,  $x_a + x_b + x_c = 1$ . Then we define the projection:

$$\mathcal{G} \setminus \mathcal{V}_M \ni x \mapsto \pi_M(x) = x_a \cdot a(M) + x_b \cdot b(M) + x_c \cdot c(M) \in \mathcal{G}_M,$$

where  $a(M)$ ,  $b(M)$ ,  $c(M)$  are the vertices of the triangle  $\mathcal{G}_M$  with corresponding labels  $a, b, c$ . When  $x \in \mathcal{V}_M$ , then  $x$  itself has a label assigned and it then mapped onto the corresponding vertex of  $\mathcal{G}_M$ .

## 2.2 Subordinate Brownian motions and their Schrödinger perturbations

Let  $\mathcal{G}^*$  be the two-sided infinite gasket, i.e. the set  $\mathcal{G}^* = \mathcal{G} \cup i(\mathcal{G})$ , where  $i$  is the reflection of  $\mathbb{R}^2$  with respect to the  $y$ -axis. Denote by  $\tilde{Z} = (\tilde{Z}_t, \mathbf{P}_x)_{t \geq 0, x \in \mathcal{G}^*}$  the Brownian motion on  $\mathcal{G}^*$ , as defined in [1]. It is a strong Markov and Feller process, whose transition density with respect to the Hausdorff measure is symmetric in its space variables, continuous, and fulfils the following subgaussian estimates:

$$c_{2.3} t^{-d_s/2} e^{-c_{2.4} \left( \frac{d(x,y)}{t^{1/d_w}} \right)^{d_w/(d_w-1)}} \leq \tilde{g}(t, x, y) \leq c_{2.5} t^{-d_s/2} e^{-c_{2.6} \left( \frac{d(x,y)}{t^{1/d_w}} \right)^{d_w/(d_w-1)}}, \quad (2.3)$$

$$x, y \in \mathcal{G}^*, \quad t > 0,$$

with positive constants  $c_{2.3}, \dots, c_{2.6}$ . By  $Z$  we denote the Brownian motion on the one-sided Sierpiński gasket  $\mathcal{G}$ , which is obtained from  $\tilde{Z}$  by the projection  $\mathcal{G}^* \rightarrow \mathcal{G}$ . One can directly check that its transition densities are given by  $g(t, x, y) = \tilde{g}(t, x, y) + \tilde{g}(t, x, i(y))$  for  $x \neq 0$  and twice this quantity when  $x = 0$ . The functions  $g$  share most of the properties of  $\tilde{g}$ , including continuity and the subgaussian estimates, with possibly worse constants  $c_{2.3} - c_{2.6}$ . We stick to the estimate (2.3) for  $g(t, x, y)$  as well.

Let  $S = (S_t, \mathbf{P})_{t \geq 0}$  be a subordinator, i.e. an increasing Lévy process taking values in  $[0, \infty]$  with  $S_0 = 0$ . Denote  $\eta_t(du) = \mathbf{P}(S_t \in du)$ ,  $t \geq 0$ . As usual, if the measures  $\eta_t(du)$  are absolutely continuous with respect to the Lebesgue measure, then the corresponding densities are also denoted by  $\eta_t(u)$ . The law of the subordinator  $S$  is uniquely determined by its Laplace transform  $\int_0^\infty e^{-\lambda s} \eta_t(ds) = e^{-t\phi(\lambda)}$ ,  $\lambda > 0$ . The function  $\phi : (0, \infty) \rightarrow [0, \infty)$  is called the Laplace exponent of  $S$  and can be represented as

$$\phi(\lambda) = b\lambda + \psi(\lambda) \quad \text{with} \quad \psi(\lambda) = \int_0^\infty (1 - e^{-\lambda s}) \rho(ds), \quad (2.4)$$

where  $b \geq 0$  is called the drift term and  $\rho$ , called the Lévy measure of  $S$ , is a  $\sigma$ -finite measure on  $(0, \infty)$  satisfying  $\int_0^\infty (s \wedge 1) \rho(ds) < \infty$ . It is well known that  $\phi$  is a Bernstein function. For more details of subordinators and Bernstein functions we refer the reader to [2, 22].

We always assume that  $Z$  and  $S$  are independent. The process  $X = (X_t, \mathbf{P}_x)_{t \geq 0, x \in \mathcal{G}}$  given by

$$X_t := Z_{S_t}, \quad t \geq 0,$$

is called the subordinate Brownian motion on  $\mathcal{G}$  (via subordinator  $S$ ). It is a symmetric Markov process having càdlàg paths. Its natural filtration is always assumed to fulfil the usual conditions.

For the entire paper, we make the following assumptions **(S1)** and **(S2)** on the subordinator.

**(S1)**  $\forall t < 0$  one has  $\eta_t(\{0\}) = 0$  and  $\int_{0+}^\infty \frac{1}{u^{d_s/2}} \eta_t(du) =: c_{2.7}(t) < \infty$ ,

**(S2)**  $\forall t < 0$  one has  $\int_1^\infty \eta_t(u, \infty) \frac{du}{u} < \infty$ .

Assumptions **(S1)**–**(S2)** are satisfied by a wide class of subordinators. Observe that whenever  $S_t = ct$  and  $X_t = Z_{ct}$  for some  $c > 0$ , i.e.,  $X$  is the time-rescaled Brownian motion with  $\eta_t(du) = \delta_{ct}(du)$  and  $\phi(\lambda) = c\lambda$ , they immediately hold. Some other specific examples including jump processes and sufficient conditions for them are discussed in [11, Remark 2.1, Lemma 2.2 and Example 2.1]. See also Section 6 at the end of this paper. For further examples we refer the reader to [22].

In the sequel we will need the following estimate of the tail of the subordinator  $S$  at infinity. It can also be used to verify **(S2)** for given  $\phi$ .

**Lemma 2.1** *Let  $S$  be a subordinator with Laplace exponent  $\phi$  such that  $\mathbf{P}[S_t = 0] = 0$ , for every  $t > 0$ . Then we have*

$$\eta_t(A, \infty) \leq \frac{t}{1 - e^{-1}} \int_0^{\frac{1}{A}} \frac{\phi(\lambda)}{\lambda} d\lambda, \quad t > 0, \quad A > 0.$$

**Proof.** Standard arguments as in [11, Lemma 2.2], yield

$$\int_0^\infty e^{-\lambda u} \eta_t(u, \infty) du \leq \frac{t\phi(\lambda)}{\lambda}, \quad \lambda > 0, \quad t > 0,$$

which, by monotonicity, leads to

$$e^{-\lambda A} A \eta_t(A, \infty) du \leq \int_0^A e^{-\lambda u} \eta_t(u, \infty) du \leq \frac{t\phi(\lambda)}{\lambda}, \quad \lambda > 0, \quad t > 0, \quad A > 0.$$

By integrating both sides of the above inequality with respect to  $\lambda$  over  $(0, 1/A)$ , we thus get

$$(1 - e^{-1}) \eta_t(A, \infty) \leq t \int_0^{\frac{1}{A}} \frac{\phi(\lambda)}{\lambda} d\lambda, \quad \lambda > 0, \quad t > 0, \quad A > 0,$$

which is the claimed inequality.  $\square$

By the first part of **(S1)**, the process  $X$  has symmetric and strictly positive transition densities given by

$$p(t, x, y) = \int_0^\infty g(u, x, y) \eta_t(du), \quad t > 0, \quad x, y \in \mathcal{G}, \quad (2.5)$$

while the second part guarantees that

$$\sup_{x, y \in \mathcal{G}} p(t, x, y) \leq c_{2.8}(t) < \infty \quad \text{for every } t > 0 \quad \text{and} \quad c_{2.9} := \sup_{t \geq 1} c_{2.8}(t) < \infty. \quad (2.6)$$

Moreover, for each fixed  $t > 0$ ,  $p(t, \cdot, \cdot)$  is a continuous function on  $\mathcal{G} \times \mathcal{G}$ , and for each fixed  $x, y \in \mathcal{G}$ ,  $p(\cdot, x, y)$  is a continuous function on  $(0, \infty)$ . The general theory of subordination (see, e.g., [22, Chapter 12]) yields that the process  $X$  is a Feller process and, in consequence, a strong Markov process. Also, by **(S1)**, it has the strong Feller property.

For an open set  $U \subset \mathcal{G}$  by  $\tau_U := \inf \{t \geq 0 : X_t \notin U\}$  we denote the first exit time of the process  $X$  from  $U$ , while  $T_D = \tau_{D^c}$  denotes the first entrance time into the closed set  $D$ . By  $(P_t)_{t \geq 0}$  we denote the semigroup with kernel  $p(t, x, y)$ , and by  $(P_t^U)_{t \geq 0}$  – the semigroup related to the process killed on exiting an open set  $U$ ;  $\lambda_1(U)$  is the principal eigenvalue of its generator. Finally, by  $\mathbf{P}_{x,y}^t$  we denote the bridge measures corresponding to process  $X$  on  $D([0, t], \mathcal{G})$  (for more details we refer to [11, p. 9]).

We say that a Borel function  $V : \mathcal{G} \rightarrow \mathbb{R}$  is in Kato class  $\mathcal{K}^X$  related to the process  $X$  if

$$\limsup_{t \searrow 0} \sup_{x \in \mathcal{G}} \int_0^t \mathbf{E}_x |V(X_s)| ds = 0. \quad (2.7)$$

Also,  $V \in \mathcal{K}_{loc}^X$  (local Kato class), when  $\mathbf{1}_B V \in \mathcal{K}^X$  for every ball  $B \subset \mathcal{G}$ . One can show that  $L_{loc}^\infty(\mathcal{G}) \subset \mathcal{K}_{loc}^X \subset L_{loc}^1(\mathcal{G}, m)$ .

In this paper, we study the subordinate Brownian motions on  $\mathcal{G}$  perturbed by random Schrödinger potentials of the Poissonian type, which are defined by

$$V(x, \omega) := \int_{\mathcal{G}} W(x, y) \mu^\omega(dy), \quad x \in \mathcal{G}, \quad \omega \in \Omega, \quad (2.8)$$

where  $\mu^\omega$  is the random counting measure corresponding to the Poisson point process on  $\mathcal{G}$ , with intensity  $\nu dm$ ,  $\nu > 0$ , defined on some probability space  $(\Omega, \mathcal{M}, \mathbb{Q})$ , and  $W : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}_+$  is a measurable, nonnegative profile function. We note for later use that for any measurable function  $f : \mathcal{G} \rightarrow \mathbb{R}_+$  one has

$$\mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_{\mathcal{G}} f(y) \mu^\omega(dy)} \right] = e^{-\int_{\mathcal{G}} \nu(1-e^{-f(y)})m(dy)}. \quad (2.9)$$

Throughout the paper we assume that the Poisson process and the Markov process  $X$  are independent and impose the following regularity assumptions on the profile function.

**(W1)**  $W \geq 0$ ,  $W(\cdot, y) \in \mathcal{K}_{loc}^X$  for every  $y \in \mathcal{G}$  and there exists a function  $h \in L^1(\mathcal{G}, m)$  such that  $W(x, y) \leq h(y)$ , whenever  $d(y, 0) \geq 2d(x, 0)$ .

**(W2)**  $\sum_{M=1}^{\infty} \sup_{x \in \mathcal{G}} \int_{B(x, 2^{M/4})^c} W(x, y) dm(y) < \infty$

**(W3)** there is  $M_0 \in \mathbb{Z}$  such that

$$\sum_{y' \in \pi_M^{-1}(\pi_M(y))} W(\pi_M(x), y') \leq \sum_{y' \in \pi_{M+1}^{-1}(\pi_M(y))} W(\pi_{M+1}(x), y'), \quad x, y \in \mathcal{G}, \quad (2.10)$$

for every  $M \in \mathbb{Z}$ ,  $M \geq M_0$ .

The conditions **(W1)**–**(W3)** have been recently introduced in [11]. Under **(W1)** we have that  $V(\cdot, \omega) \in \mathcal{K}_{loc}^X$  for  $\mathbb{Q}$ -almost all  $\omega \in \Omega$ , while the remaining conditions **(W2)**–**(W3)** guarantee sufficient regularity, needed to study the spectral problem that we address in the present paper. For discussion of the above assumptions we refer the reader to [11, Subsection 2.3.2].

For the Poissonian random potential  $V$ , we consider the Feynman-Kac semigroups  $(T_t^{D,M,\omega})_{t \geq 0}$  related to the process killed outside  $\mathcal{G}_M$ ,  $M \in \mathbb{Z}$ , consisting of the operators:

$$T_t^{D,M,\omega} f(x) = \mathbf{E}_x \left[ e^{-\int_0^t V(X_s, \omega) ds} f(X_t); t < \tau_{\mathcal{G}_M} \right], \quad f \in L^2(\mathcal{G}_M, m), \quad M \in \mathbb{Z}, \quad t > 0. \quad (2.11)$$

For  $\mathbb{Q}$ -almost all  $\omega \in \Omega$  and every  $t > 0$ ,  $T_t^{D,M,\omega}$  are symmetric, ultracontractive and Hilbert-Schmidt operators admitting measurable, symmetric and bounded kernels  $u_D^M(t, x, y)$  which are known to have the following very useful bridge representation [11, (2.27)]

$$u_D^M(t, x, y) = p(t, x, y) \mathbf{E}_{x,y}^t \left[ e^{-\int_0^t V(X_s, \omega) ds}; t < \tau_{\mathcal{G}_M} \right], \quad M \in \mathbb{Z}, \quad x, y \in \mathcal{G}_M, \quad t > 0. \quad (2.12)$$

Denote by  $A^{D,M,\omega}$  the  $L^2(\mathcal{G}, m)$ -generator of the semigroup  $(T_t^{D,M,\omega})_{t \geq 0}$ . By analogy to the Euclidean case, the operator  $-A^{D,M,\omega}$  is called the *generalized Schrödinger operator* corresponding to the generator of the process  $X$  with Dirichlet (outer) conditions. The spectrum of  $-A^{D,M,\omega}$  is purely discrete. The corresponding eigenvalues can be ordered as  $0 \leq \lambda_1^{D,M}(\omega) \leq \lambda_2^{D,M}(\omega) \leq \dots \rightarrow \infty$ . For discussion and verification of the properties and facts listed above we refer the reader to [11, Subsection 2.3.1 and 2.3.2].

The basic objects we consider are the random empirical measures on  $\mathbb{R}_+ := [0, \infty)$  based on the spectra of  $-A^{D,M,\omega}$ , normalized by the volume of  $\mathcal{G}_M$ :

$$l_M^D(\omega) := \frac{1}{m(\mathcal{G}_M)} \sum_{n=1}^{\infty} \delta_{\lambda_n^{D,M}(\omega)}, \quad M \in \mathbb{Z}_+, \quad (2.13)$$

and their Laplace transforms  $L_M^D(t, \omega) := \int_0^\infty e^{-\lambda t} dl_M^D(\omega)(t) = \frac{1}{m(\mathcal{G}_M)} \sum_{n=1}^{\infty} e^{-\lambda_n^{D,M}(\omega)t}$  which have the following representation

$$L_M^D(t, \omega) = \frac{1}{m(\mathcal{G}_M)} \text{tr} T_t^{D,M,\omega} = \frac{1}{m(\mathcal{G}_M)} \int_{\mathcal{G}_M} p(t, x, x) \mathbf{E}_{x,x}^t \left[ e^{-\int_0^t V(X_s, \omega) ds}; t < \tau_{\mathcal{G}_M} \right] m(dx).$$

We have recently proven the following result on the convergence of  $L_M^D$  and  $l_M^D$  as  $M \rightarrow \infty$ .



**Theorem 2.1** [11, Theorems 3.1 and 3.2] *Let  $S$  be a subordinator satisfying the assumptions **(S1)**–**(S2)** and let  $V$  be a Poissonian random field with the profile  $W$  satisfying the conditions **(W1)**–**(W3)**. Then for every  $t > 0$ ,  $\mathbb{E}_{\mathbb{Q}}[L_M^D(t, \omega)]$  converges to a finite limit  $L(t)$  as  $M \rightarrow \infty$ . Moreover,  $\mathbb{Q}$ –almost surely, the random measures  $l_M^D(\omega)$  vaguely converge to a common nonrandom limit measure  $l$  on  $\mathbb{R}_+$ , with Laplace transform  $L(t)$ .*

The deterministic measure  $l$  given by the above theorem is called the *integrated density of states* (IDS) for the process  $X$  perturbed by the Poissonian potential  $V$  on  $\mathcal{G}$ . The present paper is devoted to study of the limiting behaviour of  $l[0, \lambda)$  when  $\lambda \rightarrow 0^+$ .

As stated above, the assumptions **(S1)**–**(S2)** and **(W1)**–**(W3)** guarantee the existence of IDS in our settings. In present paper, the conditions **(S1)**–**(S2)** only give the general framework for our study. From now on, we will restrict our attention to the class of the so-called *complete* subordinators and impose some additional regularity on the corresponding Laplace exponent  $\phi$ . Recall that the subordinator  $S$  is called *complete* if its Laplace exponents  $\phi$  is *complete Bernstein function*, i.e., the corresponding Lévy measure  $\nu$  is absolutely continuous with respect to the Lebesgue measure with completely monotone density (see e.g. [22, Chapter 6]).

### 3 Lower bounds

#### 3.1 Lower bounds for the integrated density of states

As explained above, the integrated density of states  $l$  is the vague limit of the empirical measures based on the Laplacians on  $\mathcal{G}_M$  with Dirichlet boundary conditions, and its Laplace transform  $L(t)$  for any given  $t > 0$  can be expressed as the limit:

$$L(t) = \lim_{M \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[L_M^D(t, \omega)], \quad (3.1)$$

where

$$L_M^D(t, \omega) = \frac{1}{m(\mathcal{G}_M)} \int_{\mathcal{G}_M} p(t, x, x) \mathbf{E}_{x,x}^t \left[ e^{-\int_0^t V(X_s, \omega) ds} \mathbf{1}_{\{\tau_{\mathcal{G}_M} > t\}} \right] dm(x).$$

Before we proceed, we introduce some notation. For a gasket triangle  $\Delta$ , let us define

$$L^\Delta(t, \omega) = \frac{1}{m(\Delta)} \int_{\Delta} p(t, x, x) \mathbf{E}_{x,x}^t \left[ e^{-\int_0^t V(X_s, \omega) ds} \mathbf{1}_{\{\tau_{\Delta} > t\}} \right] dm(x), \quad (3.2)$$

so that  $L_M^D = L^{\mathcal{G}_M}$ .

In this section, we will work under the additional assumption that the Laplace exponent  $\phi$  of the subordinator  $S$  is a complete Bernstein function satisfying the following condition:

**(L1)** there exist constants  $c_{3.1} > 0$ ,  $\beta \in (0, d_w]$  and  $s_0 > 0$  such that for  $s \in (0, s_0]$  one has  $\phi(s) \leq c_{3.1} s^{\beta/d_w}$ .

Under **(L1)**, the assumption **(S2)** is automatically satisfied (it follows e.g. from Lemma 2.1).

We have the following lower bound.

**Theorem 3.1** *Let  $X$  be a subordinate Brownian motion on  $\mathcal{G}$  via a complete subordinator  $S$  with Laplace exponent  $\phi$  such that **(S1)** and **(L1)** hold and let  $V$  be a Poissonian potential with profile  $W$  satisfying **(W1)**–**(W3)**. Suppose that there exist constants  $\theta > 0$ ,  $K \in [0, \infty)$  such that*

$$\limsup_{d(x,y) \rightarrow \infty} W(x, y) d(x, y)^{d+\theta} = K. \quad (3.3)$$

Then there exist positive constants  $C_1, C_2$  and  $t_0 > 0$  such that for  $t > t_0$  one has

$$L(t) \geq \exp \left\{ -C_1 t^{d/(d+\beta)} \nu^{\beta/(d+\beta)} - C_2 t^{d/(d+\theta)} \nu \right\}. \quad (3.4)$$

In particular,

(i) when  $\beta < \theta$  then

$$\liminf_{t \rightarrow \infty} \frac{\log L(t)}{t^{d/(d+\beta)}} \geq -C_1 \nu^{\beta/(d+\beta)},$$

(ii) when  $\beta = \theta$  then

$$\liminf_{t \rightarrow \infty} \frac{\log L(t)}{t^{d/(d+\beta)}} \geq -C_1 \nu^{\beta/(d+\beta)} - C_2 \nu,$$

(iii) when  $\beta > \theta$  then

$$\liminf_{t \rightarrow \infty} \frac{\log L(t)}{t^{d/(d+\theta)}} \geq -C_2 \nu.$$

**Proof.** For given  $a > 0$ , let the 'short range' and the 'long range' profiles be given by

$$W_a(x, y) = W(x, y) \mathbf{1}_{\{d(x, y) \leq a\}}, \quad \text{and} \quad W^a(x, y) = W(x, y) \mathbf{1}_{\{d(x, y) > a\}}, \quad (3.5)$$

then let  $V_a, V^a$  be the Poissonian potentials based on  $W_a, W^a$ , accordingly. Moreover, for  $a > 0$  let

$$S_W(a) := \sup_{x \in \mathcal{G}} \int_{\mathcal{G}} W^a(x, y) \, dm(y) = \sup_{x \in \mathcal{G}} \int_{d(x, y) > a} W(x, y) \, dm(y).$$

We start with the following estimate, which is valid for  $W$  satisfying **(W1)**–**(W3)** and  $S$ —a complete subordinator satisfying **(S1)** and **(S2)**. We prove that for any  $t > 0, a > 0, M \in \mathbb{Z}_+$  one has

$$L(t) \geq \exp \left\{ -t\phi \left( \frac{1}{2^{Md_w}} \lambda_1^{BM}(\mathcal{G}_0) \right) - \nu t S_W(a) - \nu(2^{Md} + 9a^d) \right\}. \quad (3.6)$$

To prove (3.6), consider the expressions  $L_{M+k}^D(t, \omega) = L^{\mathcal{G}_{M+k}}(t, \omega)$ ,  $k \in \mathbb{Z}_+$ . Clearly, by (3.1), we have

$$L(t) = \lim_{k \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[L_{M+k}^D(t, \omega)].$$

For given  $k \geq 1$ , the set  $\mathcal{G}_{M+k}$  consists of  $3^k$  gasket triangles of size  $2^M$  each, with pairwise disjoint interiors. Denote them  $\Delta_1, \dots, \Delta_{3^k}$ . Because of the inclusions  $\Delta_i \subset \mathcal{G}_{M+k}$ , one has

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[L_{M+k}^D(t, \omega)] &= \frac{1}{3^{M+k}} \int_{\mathcal{G}_{M+k}} p(t, x, x) \mathbb{E}_{\mathbb{Q}} \mathbf{E}_{x,x}^t \left[ e^{-\int_0^t V(X_s, \omega) ds} \mathbf{1}_{\{\tau_{\mathcal{G}_{M+k}} > t\}} \right] dm(x) \\ &= \frac{1}{3^{M+k}} \sum_{i=1}^{3^k} \int_{\Delta_i} p(t, x, x) \mathbb{E}_{\mathbb{Q}} \mathbf{E}_{x,x}^t \left[ e^{-\int_0^t V(X_s, \omega) ds} \mathbf{1}_{\{\tau_{\mathcal{G}_{M+k}} > t\}} \right] dm(x) \\ &\geq \frac{1}{3^{M+k}} \sum_{i=1}^{3^k} \int_{\Delta_i} p(t, x, x) \mathbb{E}_{\mathbb{Q}} \mathbf{E}_{x,x}^t \left[ e^{-\int_0^t V(X_s, \omega) ds} \mathbf{1}_{\{\tau_{\Delta_i} > t\}} \right] dm(x) \\ &\geq \inf_i \mathbb{E}_{\mathbb{Q}}[L^{\Delta_i}(t, \omega)]. \end{aligned}$$

Pick now any of the  $i$ 's, say  $i_0$ , and let  $\mathcal{M}_{i_0}^a = \{\omega : \text{no Poisson points fell into } \Delta_{i_0}^a\}$ , where  $\Delta_{i_0}^a$  denotes the  $a$ -vicinity of  $\Delta_{i_0}$ . In particular,

$$\mathbb{E}_{\mathbb{Q}} L^{\Delta_{i_0}}(t, \omega) \geq \mathbb{E}_{\mathbb{Q}} \left[ L^{\Delta_{i_0}}(t, \omega) \mathbf{1}_{\mathcal{M}_{i_0}^a} \right]. \quad (3.7)$$



Observe that for every  $\omega \in \mathcal{M}_{i_0}^a$  and for a fixed trajectory of  $X_s$  starting at  $x \in \Delta_{i_0}$  and not leaving the set  $\Delta_{i_0}$  up to time  $t$  one has

$$V(X_s, \omega) = \int_{(\Delta_{i_0}^a)'} W(X_s, y) d\mu^\omega(y) = \int_{(\Delta_{i_0}^a)'} W^a(X_s, y) d\mu^\omega(y) = V^a(X_s, \omega). \quad (3.8)$$

For such a trajectory, the random Feynman-Kac functional  $e^{-\int_0^t V^a(X_s, \omega) ds}$  and the event  $\mathcal{M}_{i_0}^a$  are  $\mathbb{Q}$ -independent. Therefore, on the set  $\{\tau_{\Delta_{i_0}} > t\}$  one has:

$$\mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_0^t V(X_s, \omega) ds} \mathbf{1}_{\mathcal{M}_{i_0}^a} \right] = \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_0^t V^a(X_s, \omega) ds} \mathbf{1}_{\mathcal{M}_{i_0}^a} \right] = \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_0^t V^a(X_s, \omega) ds} \right] \cdot \mathbb{Q}[\mathcal{M}_{i_0}^a]. \quad (3.9)$$

Using the definition (3.2) of  $L^{\Delta_{i_0}}$ , then inserting (3.9) inside (3.7) we obtain:

$$\mathbb{E}_{\mathbb{Q}}[L^{\Delta_{i_0}}(t, \omega)] \geq \frac{1}{m(\Delta_{i_0})} \int_{\Delta_{i_0}} p(t, x, x) \mathbf{E}_{x,x}^t \left[ \mathbf{1}_{\{\tau_{\Delta_{i_0}} > t\}} \cdot \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_0^t V^a(X_s, \omega) ds} \right] \cdot \mathbb{Q}[\mathcal{M}_{i_0}^a] \right] dm(x), \quad (3.10)$$

moreover

$$\mathbb{Q}[\mathcal{M}_{i_0}^a] = \mathbb{Q}[\text{no Poisson points inside } \Delta_{i_0}^a] = e^{-\nu m(\Delta_{i_0}^a)} \geq e^{-\nu(2^{M^d} + 9a^d)}.$$

The exponential formula (2.9) applied to the inner integral in (3.10) gives

$$\mathbb{E}_{\mathbb{Q}} e^{-\int_0^t V^a(X_s, \omega) ds} = e^{-\nu \int_{\mathcal{G}} (1 - e^{-\int_0^t W^a(X_s, y) ds}) dm(y)}.$$

From an elementary inequality  $e^{-x} \geq 1 - x$ ,  $x \in \mathbb{R}$ , and the Fubini theorem we obtain that

$$\mathbb{E}_{\mathbb{Q}} e^{-\int_0^t V^a(X_s, \omega) ds} \geq e^{-\nu \int_{\mathcal{G}} \int_0^t W^a(X_s, y) dm(y) ds} \geq e^{-\nu t \sup_{x \in \mathcal{G}} \int_{d(x, y) > a} W(x, y) dm(y)} = e^{-\nu t S_W(a)}.$$

It follows

$$\mathbb{E}_{\mathbb{Q}} [L^{\Delta_{i_0}}(t, \omega)] \geq \left[ \frac{1}{m(\Delta_{i_0})} \int_{\Delta_{i_0}} p(t, x, x) \mathbf{P}_{x,x}^t [\tau_{\Delta_{i_0}} > t] dm(x) \right] \cdot e^{-\nu t S_W(a)} \cdot e^{-\nu(2^{M^d} + 9a^d)}$$

The first multiplier in the expression above is the averaged trace of the operator  $P_t^{\Delta_{i_0}}$ , and as such is not bigger than  $e^{-t\lambda_1(\Delta_{i_0})}$ . From [6, Theorem 3.4] we have  $\lambda_1(\Delta_{i_0}) \leq \phi(\lambda_1^{BM}(\Delta_{i_0}))$ , where  $\lambda_1^{BM}(U)$  denotes the principal Dirichlet eigenvalue of the Brownian motion on  $U$ . As the Brownian motions on  $\Delta_{i_0}$  and on  $\mathcal{G}_M$  are indistinguishable up to respective exit times of  $\Delta_{i_0}$ ,  $\mathcal{G}_M$ , one has  $\lambda_1^{BM}(\Delta_{i_0}) = \lambda_1^{BM}(\mathcal{G}_M)$ , and from the Brownian scaling we have  $\lambda_1^{BM}(\mathcal{G}_M) = \lambda_1^{BM}(2^M \mathcal{G}_0) = \frac{1}{2^{M d_w}} \lambda_1^{BM}(\mathcal{G}_0)$ . Collecting all the estimates above we obtain the statement

$$\mathbb{E}_{\mathbb{Q}}[L_{M+k}^D(t, \omega)] \geq \exp \left\{ -t\phi \left( \frac{1}{2^{M d_w}} \lambda_1^{BM}(\mathcal{G}_0) \right) - \nu t S_W(a) - \nu(2^{M^d} + 9a^d) \right\},$$

which after letting  $k$  go to infinity gives (3.6).

Having proven (3.6), we will now use the remaining assumptions. From **(L1)** we get

$$\phi \left( \frac{1}{2^{M d_w}} \lambda_1^{BM}(\mathcal{G}_0) \right) \leq \frac{c_{3.1}}{2^{M \beta}} \cdot (\lambda_1^{BM}(\mathcal{G}_0))^{\beta/d_w} =: \frac{c_{3.2}}{2^{M \beta}}, \quad (3.11)$$

for  $M$  large enough.

The condition (3.3) combined with (2.2) permit us to write

$$\int_{d(x,y)>a} W(x,y) dm(y) \leq \int_{d(x,y)>a} \frac{(K+o(1))}{d(x,y)^{d+\theta}} dm(y) \leq c_{2.2}(K+o(1)) \frac{1}{a^\theta}, \quad (3.12)$$

and consequently

$$S_W(a) \leq c_{2.2}(K+o(1)) \frac{1}{a^\theta}, \quad \text{as } a \rightarrow \infty. \quad (3.13)$$

Next, for given  $t > 0$ , choose  $M = M(t)$  to be the unique integer satisfying

$$2^M \leq \left(\frac{t}{\nu}\right)^{\frac{1}{d+\beta}} < 2^{M+1}. \quad (3.14)$$

Inserting (3.11), (3.13), and (3.14) into (3.6), and using  $a = t^{1/(d+\theta)}$ , after some elementary algebra we obtain

$$L(t) \geq \exp \left\{ - (c_{3.2} + 1) t^{d/(d+\beta)} \nu^{\beta/(d+\beta)} - (Kc_{2.2} + 9 + o(1)) t^{d/(d+\theta)} \nu \right\}, \quad \text{as } t \rightarrow \infty.$$

To get (3.4), we just set  $C_1 = (c_{3.2} + 1)$ ,  $C_2 = (Kc_{2.2} + 9)$ . Statements (i)–(iii) are straightforward consequences of (3.4).  $\square$

### 3.2 Lower bounds for the Feynman-Kac functional

The methods we use are also suitable for obtaining bounds on the averaged Feynman-Kac functional, i.e.  $\mathbb{E}_{\mathbb{Q}} \mathbf{E}_x \left[ e^{-\int_0^t V(X_s, \omega) ds} \right]$ . In this case, the assumptions concerning the process and the profile  $W$  can be somewhat relaxed.

**Theorem 3.2** *Let  $X$  be a subordinate Brownian motion on  $\mathcal{G}$  via a complete subordinator  $S$  with Laplace exponent  $\phi$  such that (S1) and (L1) hold and let the potential profile  $W$  fulfil (W1) and (3.3). Then, with constants  $C_1, C_2 > 0$  from Theorem 3.1, for any  $x \in \mathcal{G}$  one has:*

(i) *when  $\beta < \theta$  then*

$$\liminf_{t \rightarrow \infty} \frac{\log \mathbb{E}_{\mathbb{Q}} \mathbf{E}_x [e^{-\int_0^t V(X_s, \omega) ds}]}{t^{d/(d+\beta)}} \geq -C_1 \nu^{\beta/(d+\beta)},$$

(ii) *when  $\beta = \theta$  then*

$$\liminf_{t \rightarrow \infty} \frac{\log \mathbb{E}_{\mathbb{Q}} \mathbf{E}_x [e^{-\int_0^t V(X_s, \omega) ds}]}{t^{d/(d+\beta)}} \geq -C_1 \nu^{\beta/(d+\beta)} - C_2 \nu,$$

(iii) *when  $\beta > \theta$  then*

$$\liminf_{t \rightarrow \infty} \frac{\log \mathbb{E}_{\mathbb{Q}} \mathbf{E}_x [e^{-\int_0^t V(X_s, \omega) ds}]}{t^{d/(d+\theta)}} \geq -C_2 \nu.$$

**Proof.** The proofs of these statements are very much alike those from Theorem 3.1. Fix  $M \in \mathbb{Z}_+$ ,  $a > 0$ ,  $t > 0$ ,  $x \in \mathcal{G}$ , for the moment assuming only that  $M$  is so large that  $x \in \text{Int } \mathcal{G}_M$ . Let  $\mathcal{M}_M^a$  be the event ‘no Poisson points fell into  $\mathcal{G}_M^a$ ’. Using the reasoning that led to (3.9), we get that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \mathbf{E}_x \left[ e^{-\int_0^t V(X_s, \omega) ds} \right] &\geq \mathbb{E}_{\mathbb{Q}} \mathbf{E}_x \left[ e^{-\int_0^t V(X_s, \omega) ds} \mathbf{1}_{\{\tau_{\mathcal{G}_M} > t\}} \mathbf{1}_{\mathcal{M}_M^a} \right] \\ &\geq \mathbf{E}_x \left[ \mathbf{1}_{\{\tau_{\mathcal{G}_M} > t\}} \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_0^t V^a(X_s, \omega) ds} \right] \right] \mathbb{Q}[\mathcal{M}_M^a] \\ &\geq \mathbf{P}_x[\tau_{\mathcal{G}_M} > t] \cdot e^{-\nu t S_W(a) - \nu(2^{Md} + 9a^d)}. \end{aligned}$$

The probability space  $(\Omega, \mathcal{F}, \mathbf{P}_x)$  for  $X$  can be realized as a product of two probability spaces on which  $Z$  and  $S$  are defined. In particular,  $\mathbf{P}_x = \mathbf{P}_x^X := \mathbf{P}_x^Z \otimes \mathbf{P}$ . Therefore, it is easy to observe that for every  $M \in \mathbb{Z}_+$ ,  $x \in \mathcal{G}_M$ , and  $t > 0$ , by Fubini we have

$$\mathbf{P}_x^X[\tau_{\mathcal{G}_M}^X > t] \geq \mathbf{P}_x^Z \otimes \mathbf{P}[\tau_{\mathcal{G}_M}^Z > S_t] = \int_0^\infty \mathbf{P}_x^Z[\tau_{\mathcal{G}_M}^Z > u] \eta_t(du).$$

By the scaling of  $Z$ , we have  $\mathbf{P}_x^Z[\tau_{\mathcal{G}_M}^Z > u] = \mathbf{P}_{(x/2^M)}^Z[\tau_{\mathcal{G}_0}^Z > 2^{-Mdw}u]$ . Moreover, one has

$$c\mathbf{P}_x^Z[\tau_{\mathcal{G}_0}^Z > u] \geq \mathbf{P}_x^Z[\tau_{\mathcal{G}_0}^Z > u; \varphi_1^{0,Z}(Z_t)] = e^{-u\lambda_1^{BM}(\mathcal{G}_0)} \varphi_1^{0,Z}(x), \quad x \in \mathcal{G}_0, \quad u > 0,$$

where  $\lambda_1^{BM}(\mathcal{G}_0)$  is the principal eigenvalue of the Brownian motion on  $\mathcal{G}_0$  with killing on  $\partial\mathcal{G}_0$ ,  $\varphi_1^{0,Z}$  is the corresponding normalized eigenfunction, and  $c = \|\varphi_1^{0,Z}\|_\infty < \infty$  is independent of  $M$  and  $u$ . The transition density of the process  $Z$  killed on exiting  $\mathcal{G}_0$  is positive for all  $u > 0$ ,  $x, y \in \text{Int } \mathcal{G}_0$ , thus from general theory its ground state is continuous and can be chosen to be strictly positive on  $\overline{B}(0, \frac{1}{2})$  (note that  $0 \notin \partial\mathcal{G}_0 \subset \mathcal{G}$ ). Denote  $c^{(1)} = \inf_{y \in B(0, 1/2)} \varphi_1^{0,Z}(y) > 0$  and  $c^{(2)} = c^{-1} \cdot c^{(1)}$ . Collecting all the above estimates, we get

$$\mathbf{P}_x^X[\tau_{\mathcal{G}_M}^X > t] \geq \int_0^\infty \mathbf{P}_{(x/2^M)}^Z[\tau_{\mathcal{G}_0}^Z > 2^{-Mdw}u] \eta_t(du) \geq c^{(2)} \int_0^\infty e^{-2^{-Mdw}u\lambda_1^{BM}(\mathcal{G}_0)} \eta_t(du),$$

whenever  $x \in \mathcal{G}_M/2$ , i.e.,  $x/2^M \in B(0, 1/2)$ . Observe the last integral is the Laplace transform of  $\eta_t$ . Therefore,

$$\mathbf{P}_x^X[\tau_{\mathcal{G}_M}^X > t] \geq c^{(2)} e^{-t\phi(2^{-Mdw}\lambda_1^{BM}(\mathcal{G}_0))}.$$

Using now the condition **(L1)**, we finally obtain

$$\mathbf{P}_x^X[\tau_{\mathcal{G}_M}^X > t] \geq c^{(2)} e^{-c_{3.1}t \cdot 2^{-M\beta dw}}.$$

All these arguments were true for any  $M$ , as long as  $x \in \mathcal{G}_M/2$ . At this point we declare the specific value of  $M$ : we take it equal to  $M(t)$  given by (3.14). Moreover, choose again  $a = t^{\frac{1}{d+\theta}}$ . For this choice of  $M, a$ , identically as before we obtain, as long as  $x \in \mathcal{G}_{M(t)}/2$  and  $t \rightarrow \infty$ ,

$$\mathbb{E}_{\mathbb{Q}} \mathbf{E}_x \left[ e^{-\int_0^t V(X_s, \omega) ds} \right] \geq c^{(2)} \exp \left\{ - (c_{3.2} + 1) t^{d/(d+\beta)} \nu^{\beta/(d+\beta)} - (Kc_{2.2} + 9 + o(1)) t^{d/(d+\theta)} \nu \right\}.$$

From this inequality all the statements follow as before. □

## 4 Upper bounds

### 4.1 Upper bound for the long range interaction

We first derive the upper bound which depends only on the long range behaviour of the potential. It does not require any additional assumptions on the subordinator  $S$ . The following result is useful for profile functions  $W$  with slow decay at infinity.

**Proposition 4.1** *Let  $X$  be a subordinate Brownian motion on  $\mathcal{G}$  via a subordinator  $S$  satisfying **(S1)**-**(S2)** and let  $V$  be a Poissonian potential with profile  $W$  such that the assumptions **(W1)**-**(W3)** hold. Then for every  $t \geq 1$  and  $a > 0$  we have*

$$L(t) \leq c_{2.9} e^{-\nu R_W(a, t)}, \tag{4.1}$$

where  $R_W(a, t) := \inf_{x \in \mathcal{G}} \int_{d(x, y) > a} (1 - e^{-tW(x, y)}) m(dy)$ , and  $c_{2.9}$  is the constant from (2.6). In particular, if for a number  $\theta > 0$  there is  $K \in [0, \infty)$  such that

$$\liminf_{d(x, y) \rightarrow \infty} W(x, y) d(x, y)^{d+\theta} = K, \quad (4.2)$$

then we have

$$\limsup_{t \rightarrow \infty} \frac{\log L(t)}{t^{\frac{d}{d+\theta}}} \leq -\frac{K}{c_{2.2} e^K} \nu.$$

**Proof.** Since for every  $t > 0$  we have  $L(t) = \lim_{M \rightarrow \infty} \mathbb{E}_{\mathbb{Q}} L_M^D(t, \omega)$ , it is enough to show that for every  $t \geq 1$ ,  $a > 0$  we have

$$\mathbb{E}_{\mathbb{Q}} L_M^D(t, \omega) \leq c_{2.9} e^{-\nu R_W(a, t)}.$$

Recall that for every  $t \geq 1$  and  $x, y \in \mathcal{G}$  we have  $p(t, x, y) \leq c_{2.9}$  (see (2.6)). By this bound and the exponential formula (2.9), we thus get

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} L_M^D(t, \omega) &\leq \frac{c_{2.9}}{m(\mathcal{G}_M)} \int_{\mathcal{G}_M} \mathbf{E}_{x, x}^t \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_0^t V(X_s, \omega) ds} \right] m(dx) \\ &= \frac{c_{2.9}}{m(\mathcal{G}_M)} \int_{\mathcal{G}_M} \mathbf{E}_{x, x}^t \left[ e^{-\nu \int_{\mathcal{G}} (1 - e^{-\int_0^t W(X_s, y) ds}) m(dy)} \right] m(dx). \end{aligned}$$

The integral over  $\mathcal{G}$  in the exponent can be written as

$$\int_{\mathcal{G}} \left( 1 - e^{-\int_0^t W(X_s, y) ds} \right) m(dy) = \int_{\mathcal{G}} F \left( \int_0^t t W(X_s, y) \frac{ds}{t} \right) m(dy),$$

where the function  $F$  is given by  $F(t) = 1 - e^{-t}$ . It is a concave function, therefore from Jensen's inequality for concave functions and Fubini's theorem we get

$$\begin{aligned} \int_{\mathcal{G}} \left( 1 - e^{-\int_0^t W(X_s, y) ds} \right) m(dy) &= \int_{\mathcal{G}} F \left( \int_0^t t W(X_s, y) \frac{ds}{t} \right) m(dy) \\ &\geq \int_{\mathcal{G}} \int_0^t F(t W(X_s, y)) \frac{ds}{t} m(dy) \\ &= \int_0^t \int_{\mathcal{G}} \left( 1 - e^{-t W(X_s, y)} \right) m(dy) \frac{ds}{t} \\ &\geq \inf_{x \in \mathcal{G}} \int_{\mathcal{G}} \left( 1 - e^{-t W(x, y)} \right) m(dy). \end{aligned}$$

In particular, for all  $t \geq 1$ ,  $a > 0$  and  $M \in \mathbb{Z}_+$ , we obtain

$$\mathbb{E}_{\mathbb{Q}}[L_M^D(t, \omega)] \leq c_{2.9} e^{-\nu \inf_{x \in \mathcal{G}} \int_{d(x, y) > a} (1 - e^{-t W(x, y)}) m(dy)} = c_{2.9} e^{-\nu R_W(a, t)},$$

which completes the proof of (4.1).

To show the second assertion, first note that by the standard inequality  $1 - e^{-s} \geq e^{-s}s$ ,  $s \geq 0$ , and (4.2), for  $d(x, y) > a$  and  $t > 0$ , we have

$$1 - e^{-tW(x, y)} \geq 1 - e^{-t(K+o(1))d(x, y)^{-d-\theta}} \geq t(K+o(1))d(x, y)^{-d-\theta} e^{-t(K+o(1))d(x, y)^{-d-\theta}} \quad \text{as } a \rightarrow \infty.$$

By taking  $a = t^{\frac{1}{d+\theta}}$  with  $t \rightarrow \infty$ , we thus get, using (2.2):

$$\begin{aligned} \int_{d(x,y) > t^{\frac{1}{d+\theta}}} \left(1 - e^{-tW(x,y)}\right) m(dy) &\geq t(K + o(1))e^{-(K+o(1))} \int_{d(x,y) > t^{\frac{1}{d+\theta}}} d(x,y)^{-d-\theta} m(dy) \\ &\geq t(K + o(1))e^{-(K+o(1))} c_{2.2}^{-1} t^{\frac{-\theta}{d+\theta}} \\ &= c_{2.2}^{-1} t^{\frac{d}{d+\theta}} (K + o(1))e^{-(K+o(1))}. \end{aligned}$$

By (4.1), we conclude that

$$\limsup_{t \rightarrow \infty} \frac{\log L(t)}{t^{d/d+\theta}} \leq -\frac{K}{c_{2.2}e^K} \nu.$$

The proof is complete. □

Similar bounds hold true for the averaged Feynman-Kac functional.

**Proposition 4.2** *Let  $X$  be a subordinate Brownian motion via the subordinator  $S$  satisfying **(S1)** – **(S2)** and let  $V$  be a Poissonian potential with profile  $W$  satisfying **(W1)**. Then for every  $t \geq 1$  and  $a > 0$  we have*

$$\sup_{x \in \mathcal{G}} \mathbb{E}_{\mathbb{Q}} \mathbf{E}_x \left[ e^{-\int_0^t V(X_s, \omega) ds} \right] \leq e^{-\nu R_W(a, t)}. \quad (4.3)$$

In particular, if (4.2) holds with some  $\theta > 0$  and  $K \in [0, \infty)$ , then for every  $x \in \mathcal{G}$  we have

$$\limsup_{t \rightarrow \infty} \frac{\log \left( \mathbb{E}_{\mathbb{Q}} \mathbf{E}_x \left[ e^{-\int_0^t V(X_s, \omega) ds} \right] \right)}{t^{d/d+\theta}} \leq -\frac{K}{c_{2.2}e^K} \nu.$$

**Proof.** For every  $x \in \mathcal{G}$ , we have

$$\mathbb{E}_{\mathbb{Q}} \mathbf{E}_x \left[ e^{-\int_0^t V(X_s) ds} \right] = \mathbf{E}_x \left[ e^{-\nu \int_{\mathcal{G}} \left(1 - e^{-\int_0^t W(X_s, y) ds}\right) m(dy)} \right] \leq e^{-\nu R_W(a, t)}, \quad a > 0,$$

and the second assertion follows exactly by the same argument as in Proposition 4.1. □

## 4.2 Upper bound for the short range interaction

Recall that we assume the Laplace exponent  $\phi$  to be a complete Bernstein function of the form

$$\phi(\lambda) = b\lambda + \psi(\lambda) \quad \text{with} \quad \psi(\lambda) = \int_0^\infty (1 - e^{-\lambda u}) \rho(u) du, \quad \lambda \geq 0. \quad (4.4)$$

In this subsection we need stronger assumptions on the Laplace exponent  $\phi$ :

**(U1)**  $b > 0$  and  $\psi \equiv 0$  (equivalently,  $\nu \equiv 0$ ; no jumps)

or

**(U2)**  $b > 0$  and  $\psi \neq 0$  satisfies the following weak scaling conditions: there are  $\alpha_1, \alpha_2, \beta, \delta \in (0, d_w)$ ,  $a_1, a_2 \in (0, 1]$ ,  $a_3, a_4 \in [1, \infty)$  and  $r_0 > 0$  such that

$$a_1 \lambda^{\alpha_1/d_w} \psi(r) \leq \psi(\lambda r) \leq a_3 \lambda^{\beta/d_w} \psi(r), \quad \lambda \in (0, 1], \quad r \in (0, r_0] \quad (4.5)$$

$$\text{and} \quad a_2 \lambda^{\alpha_2/d_w} \psi(r) \leq \psi(\lambda r) \leq a_4 \lambda^{\delta/d_w} \psi(r), \quad \lambda \geq 1, \quad r \geq r_0 \quad (4.6)$$

or

(U3)  $b = 0$  and  $\psi \neq 0$  satisfies (4.5) and (4.6) with  $\alpha_1 = \alpha_2$ .

Note that under the assumption (U1) the subordinator  $S$  is a pure drift, while the left hand sides of (4.5) and (4.6) imply the lower bounds

$$\phi(\lambda) \geq \psi(\lambda) \geq \bar{a}_1 \lambda^{\alpha_1/d_w}, \quad \lambda \in (0, 1], \quad (4.7)$$

$$\phi(\lambda) \geq \psi(\lambda) \geq \bar{a}_2 \lambda^{\alpha_2/d_w}, \quad \lambda \in [1, \infty) \quad (4.8)$$

(we have set  $\bar{a}_i = a_i \psi(r_0) r_0^{-\alpha_i/d_w}$ ).

Moreover, one can directly check that if (U1), (U2), or (U3) is satisfied, then both assumptions (S1) and (S2) hold (see [11, Remark 2.1 (2) and Lemma 2.2]). Assumption (L1) is in this case true as well. Examples of subordinators with Laplace exponents satisfying (U1) – (U3) and the corresponding subordinate Brownian motions will be discussed in Section 6.

We will need the following estimates, which are consequences of (U2) or (U3).

**Lemma 4.1** *Let  $S$  be a complete subordinator with Laplace exponent  $\phi$  given by (4.4). Under the condition (U2) or (U3) the following estimates hold.*

(a) *There exists a constant  $c_{4.1} = c_{4.1}(\phi) \in (0, 1]$  such that*

$$\rho(s) \geq c_{4.1} s^{-1} \cdot \begin{cases} s^{-\alpha_1/d_w} & \text{if } s \geq 1, \\ s^{-\alpha_2/d_w} & \text{if } s \in (0, 1]. \end{cases}$$

(b) *There exists a constant  $c_{4.2} = c_{4.2}(\phi) > 0$  such that*

$$\int_0^\infty u^{-d_s/2} \eta_t(du) \leq c_{4.2} \left( t^{-d/\alpha_1} + t^{-d/\alpha_2} \right), \quad t > 0.$$

**Proof.** We first prove (a). By [12, Proposition 2.5] the conditions (4.5) and (4.6) imply that there is a constant  $c^{(1)} > 0$  such that

$$\rho(s) \geq c^{(1)} s^{-1} \psi(s^{-1}), \quad s > 0.$$

This, together with (4.7) and (4.8), imply the claimed inequalities in (a).

Consider now (b). We have

$$e^{-t\phi(\lambda^{2/d_s})} = \int_0^\infty e^{-\lambda^{2/d_s} u} \eta_t(du), \quad t > 0, \lambda > 0.$$

By integrating in this equality with respect to  $\lambda$  over  $(0, \infty)$  and by Fubini, we get

$$\int_0^\infty e^{-t\phi(\lambda^{2/d_s})} d\lambda = \int_0^\infty \left( \int_0^\infty e^{-(u^{d_s/2} \lambda)^{2/d_s}} d\lambda \right) \eta_t(du), \quad t > 0.$$

Now, the substitution  $\vartheta = u^{d_s/2} \lambda$  in the internal integral on the right hand side gives

$$\int_0^\infty e^{-t\phi(\lambda^{2/d_s})} d\lambda = \int_0^\infty e^{-\vartheta^{2/d_s}} d\vartheta \cdot \int_0^\infty u^{-d_s/2} \eta_t(du), \quad t > 0,$$

that is,

$$\int_0^\infty u^{-d_s/2} \eta_t(du) = \frac{1}{c^{(2)}} \int_0^\infty e^{-t\phi(\lambda^{2/d_s})} d\lambda, \quad t > 0,$$



with  $(0, \infty) \ni c^{(2)} := \int_0^\infty e^{-\vartheta^{2/d_s}} d\vartheta$ . It is enough to estimate the integral on the right hand side. Recalling that  $d_s = 2d/d_w$  and applying the bounds (4.7), (4.8),

$$\int_0^\infty e^{-t\phi(\lambda^{2/d_s})} d\lambda \leq \left( \int_0^1 e^{-\bar{a}_1 t \lambda^{\alpha_1/d}} d\lambda + \int_1^\infty e^{-\bar{a}_2 t \lambda^{\alpha_2/d}} d\lambda \right), \quad t > 0.$$

Finally, by substitution  $\vartheta = t^{d/\alpha_i} \lambda$  in respective integrals, we conclude that

$$\int_0^\infty e^{-t\phi(\lambda^{2/d_s})} d\lambda \leq c^{(3)} \left( t^{-d/\alpha_1} + t^{-d/\alpha_2} \right), \quad t > 0.$$

We set  $c_{4.2} = c^{(3)}/c^{(2)}$ . The proof is complete.  $\square$

Observe that by Lemma 4.1 (b) we have  $c_{2.7}(t) \leq c_{4.2} (t^{-d/\alpha_1} + t^{-d/\alpha_2})$ , where  $c_{2.7}(t)$  comes from (S1).

#### 4.2.1 Reflected subordinate Brownian motions and their Schrödinger perturbations

Our results in this section strongly rely on the so-called reflected subordinate Brownian motions introduced recently in [11]. Therefore first we need to make a necessary preparation. For more detail discussion and justification of all properties of reflected processes listed below we refer the reader to [11, Subsection 2.2.3] and references therein.

Let  $M \in \mathbb{Z}_+$  and let  $Z^M$  be the reflected Brownian motion in  $\mathcal{G}_M$  introduced in [19], i.e. a Feller diffusion with strictly positive transition densities with respect to  $m$ , which are given by the formula

$$g^M(t, x, y) = \begin{cases} \sum_{y' \in \pi_M^{-1}(y)} g(t, x, y'), & \text{when } x, y \in \mathcal{G}_M, y \notin \mathcal{V}_M \setminus \{0\}, \\ 2 \sum_{y' \in \pi_M^{-1}(y)} g(t, x, y'), & \text{when } x \in \mathcal{G}_M, y \in \mathcal{V}_M \setminus \{0\}, \end{cases}$$

where  $\pi_M$  is the projection described in Subsection 2.1. The function  $g^M(t, x, y)$  is jointly continuous in  $(t, x, y)$  and symmetric in its space variables. It follows from scaling properties of  $g$  and properties of the projections  $\pi_M$  that

$$g^M(t, x, y) = 2^{-Md} g^0(2^{-Md_w} t, 2^{-M} x, 2^{-M} y), \quad x, y \in \mathcal{G}_M, t > 0, M \in \mathbb{Z}_+. \quad (4.9)$$

The transition semigroup of the processes  $Z^M$  and the corresponding Dirichlet forms will be denoted by  $(G_t^M)_{t \geq 0}$  and  $(\mathcal{E}_{(d_w)}^M, \mathcal{F}_{(d_w)}^M)$ , respectively. Recall that

$$\begin{aligned} \mathcal{E}_{(d_w)}^M(u, u) &:= \lim_{t \rightarrow 0^+} \left( \frac{u - G_t^M u}{t}, u \right)_{L^2(\mathcal{G}_M, m)} \\ &= \lim_{t \rightarrow 0^+} \frac{1}{2t} \int_{\mathcal{G}_M \times \mathcal{G}_M} (u(x) - u(y))^2 g^M(t, x, y) m(dx) m(dy), \end{aligned} \quad (4.10)$$

for all functions  $u \in \mathcal{F}_{(d_w)}^M$ , i.e. for those functions for which the limit in (4.10) is finite. One can directly check that by (4.9) we have

$$\mathcal{E}_{(d_w)}^M(u, u) = 2^{-Md_w} \mathcal{E}_{(d_w)}^0(u_M, u_M) \quad \text{with} \quad u_M(\cdot) = 2^{Md/2} u(2^M \cdot), \quad M \in \mathbb{Z}_+.$$

The symmetric Markov process  $X^M = (X_t^M, \mathbf{P}_x^M)_{t \geq 0, x \in \mathcal{G}_M}$  given by  $X_t^M := Z_{S_t}^M$  is called the *reflected subordinate Brownian motion* via the subordinator  $S$  in  $\mathcal{G}_M$ . Throughout this section we always assume that the subordinator  $S$  meets one of the assumptions (U1), (U2) or (U3), which

means that also the both regularity conditions **(S1)**–**(S2)** hold. Processes  $Z^M$  and  $S$  are always assumed to be stochastically independent and, therefore, the subordination formula

$$p^M(t, x, y) = \int_0^\infty g^M(u, x, y) \eta_t(du), \quad x, y \in \mathcal{G}_M, \quad t > 0, \quad (4.11)$$

defines the transition densities of the process  $X^M$ . Kernels  $p^M$  inherit the symmetry from  $g^M$  and have the same continuity properties as  $p$ , given by (2.5). Moreover, when the assumption **(U1)** holds, we simply have  $X_t^M = Z_{bt}^M$  and  $p^M(t, x, y) = g^M(bt, x, y)$  for all  $t > 0$  and  $x, y \in \mathcal{G}_M$ , while under **(U2)** or **(U3)**,  $X^M$  is a jump process with density  $p^M$  satisfying the following upper bound (cf. [11, formula (2.13)]).

**Lemma 4.2** *Under the assumption **(U2)** or **(U3)** there is a constant  $c_{4.3} = c_{4.3}(\phi) > 0$  such that*

$$p^M(t, x, y) \leq c_{4.3} \left( (t \wedge 2^{M\beta})^{-d/\alpha_1} + (t \wedge 2^{M\beta})^{-d/\alpha_2} + (t \wedge 2^{M\beta})^{-d/\beta} \right), \quad t > 0, \quad x, y \in \mathcal{G}_M, \quad M \in \mathbb{Z}_+. \quad (4.12)$$

**Proof.** By lemma [11, Lemma 2.5, ineq. (2.13)], we have

$$g^M(u, x, y) \leq c \left( u^{-d/d_w} \vee 2^{-Md} \right), \quad u > 0, \quad x, y \in \mathcal{G}_M, \quad M \in \mathbb{Z}_+,$$

with an absolute constant  $c > 0$ . Therefore, by the subordination formula (4.11), we get

$$p^M(t, x, y) \leq c \left( \int_0^{2^{Md_w}} u^{-d_s/2} \eta_t(du) + 2^{-Md} \eta_t(2^{Md_w}, \infty) \right), \quad t > 0, \quad x, y \in \mathcal{G}_M, \quad M \in \mathbb{Z}_+.$$

Note that by Lemma 4.1 (b) the first member of the sum above is smaller than  $c^{(1)} (t^{-d/\alpha_1} + t^{-d/\alpha_2})$  for some constant  $c^{(1)} > 0$ , and all  $t > 0$  and  $M \in \mathbb{Z}_+$ . Furthermore, it immediately follows from Lemma 2.1 and the upper bound in (4.5) of **(U2)** (or **(U3)**) that

$$\eta_t(2^{Md_w}, \infty) \leq (c^{(2)} t 2^{-M\beta} \wedge 1), \quad t > 0, \quad M \in \mathbb{Z}_+.$$

Collecting both estimates above, we obtain

$$p^M(t, x, y) \leq c^{(3)} \left( t^{-d/\alpha_1} + t^{-d/\alpha_2} + 2^{-Md} (t 2^{-M\beta} \wedge 1) \right), \quad t > 0, \quad x, y \in \mathcal{G}_M, \quad M \in \mathbb{Z}_+.$$

Furthermore, when  $t \leq 2^{M\beta}$ , then one has  $2^{-Md} (t 2^{-M\beta} \wedge 1) = t 2^{-M(\beta+d)} \leq t \cdot t^{-(\beta+d)/\beta} = t^{-d/\beta}$ , while for  $t > 2^{M\beta}$  we have  $t^{-d/\alpha_i} \leq 2^{-M\beta d/\alpha_i}$ . This results in the bound (4.12).  $\square$

By  $\mathbf{P}_{x,y}^{M,t}$  we denote the bridge measures corresponding to process  $X^M$  on  $D([0, t], \mathcal{G}_M)$  (for more details we refer to [11, p. 11-12]).

The process  $X^M$  corresponding to the specific subordinator  $S$  with Laplace exponent  $\phi(\lambda) = \lambda^{\gamma/d_w}$ ,  $\gamma \in (0, d_w]$ , will be singled out below. We will denote it by  $X_{(\gamma)}^M$  and, by analogy to the Euclidean case, we call it the  $\gamma$ -stable reflected subordinate Brownian motion in  $\mathcal{G}_M$ . Clearly, when  $\gamma = d_w$ , then we just have  $X_{(\gamma)}^M = Z^M$ . Note that stable processes reflected in  $\mathcal{G}_0$  were recently considered in [14].

By  $(\mathcal{E}_\phi^M, \mathcal{F}_\phi^M)$  we denote the Dirichlet form corresponding to the reflected process  $X^M$  in  $\mathcal{G}_M$  (resp.  $(\mathcal{E}_{(\gamma)}^M, \mathcal{F}_{(\gamma)}^M)$  for  $X_{(\gamma)}^M$ ). We always have  $\mathcal{F}_{(d_w)}^M \subset \mathcal{F}_\phi^M$ . It is known (see [6, 18]) that when  $b > 0$  then  $\mathcal{F}_\phi^M = \mathcal{F}_{(d_w)}^M$ , and for  $u \in \mathcal{F}_{(d_w)}^M$  we have

$$\begin{aligned} \mathcal{E}_\phi^M(u, u) &= b\mathcal{E}_{(d_w)}^M(u, u) + \int_0^\infty (u - G_s^M u, u)_{L^2(\mathcal{G}_M, m)} \rho(s) ds \\ &= b\mathcal{E}_{(d_w)}^M(u, u) + \int_{\mathcal{G}_M \times \mathcal{G}_M} (u(x) - u(y))^2 J_\phi^M(x, y) m(dx) m(dy), \end{aligned}$$

where

$$J_\phi^M(x, y) = \frac{1}{2} \int_0^\infty g^M(s, x, y) \rho(s) ds. \quad (4.13)$$

For  $b = 0$  ( $S$  has no drift) we have

$$\mathcal{F}_\phi^M = \left\{ u \in L^2(\mathcal{G}_M, m) : \int_0^\infty (u - G_s^M u, u)_{L^2(\mathcal{G}_M, m)} \rho(s) ds < \infty \right\}$$

and for  $u \in \mathcal{F}_\phi^M$

$$\mathcal{E}_\phi^M(u, u) = \int_{\mathcal{G}_M \times \mathcal{G}_M} (u(x) - u(y))^2 J_\phi^M(x, y) m(dx) m(dy).$$

In the sequel we will investigate the process  $X^M$  perturbed by potentials  $V(x)$ ,  $x \in \mathcal{G}_M$ , such that  $V \in \mathcal{K}^{X^M}$ . Recall that the Kato class  $\mathcal{K}^{X^M}$  related to the process  $X^M$  consists of functions  $V$  satisfying the condition  $\lim_{t \rightarrow 0^+} \sup_{x \in \mathcal{G}_M} \mathbf{E}_x^M \left[ \int_0^t |V|(X_s^M) ds \right] = 0$  (one can check that if  $V \in \mathcal{K}_{loc}^X$ , then  $V \mathbf{1}_{\mathcal{G}_M} \in \mathcal{K}^{X^M}$ ). The corresponding transition semigroup, which we call the Feynman-Kac semigroup associated to the process  $X^M$  and the potential  $V$ , consists of operators

$$T_t^{\phi, V, M} f(x) = \mathbf{E}_x^M \left[ e^{-\int_0^t V(X_s^M) ds} f(X_t^M) \right], \quad f \in L^2(\mathcal{G}_M, m), \quad t > 0.$$

(for  $X_{(\gamma)}^M$ ,  $\gamma \in (0, d_w]$ , we write  $T_t^{\gamma, V, M}$ ). Again, for every  $t > 0$ , the operators  $T_t^{\phi, V, M}$  (resp.  $T_t^{\gamma, V, M}$ ) are of Hilbert-Schmidt type and have purely discrete spectrum of the form  $\{\exp(-t\lambda_n^M(\phi, V))\}_{n \geq 1}$  (resp.  $\{\exp(-t\lambda_n^M(\gamma, V))\}_{n \geq 1}$ ), such that  $0 \leq \lambda_1^M(\phi, V) < \lambda_2^M(\phi, V) \leq \lambda_3^M(\phi, V) \leq \dots \rightarrow \infty$ . For the verification of the above properties and more details on the Feynman-Kac semigroups of the reflected subordinate Brownian motions we refer the reader to [11, Subsection 2.3.1].

We also define the Dirichlet form  $(\mathcal{E}_{\phi, V}^M, \mathcal{F}_{\phi, V}^M)$  corresponding to the 'reflected' process  $X^M$  perturbed by  $V$  (resp.  $(\mathcal{E}_{(\gamma), V}^M, \mathcal{F}_{(\gamma), V}^M)$ , for  $X_{(\gamma)}^M$ ). Since  $V \in \mathcal{K}^{X^M}$ , we also have  $V \in L^1(\mathcal{G}_M, m)$  and, by general theory of Dirichlet forms [10, Section 6], it holds that

$$\mathcal{F}_{\phi, V}^M = \mathcal{F}_\phi^M \cap L^2(\mathcal{G}_M, V(x)m(dx))$$

and for  $u \in \mathcal{F}_{\phi, V}^M$  we have

$$\mathcal{E}_{\phi, V}^M(u, u) = \mathcal{E}_\phi^M(u, u) + \int_{\mathcal{G}_M} V(x) u^2(x) m(dx).$$

As above, for  $M \in \mathbb{Z}_+$  and a function  $u \in L^2(\mathcal{G}_M, m)$  we define  $u_M(x) = 2^{Md/2} u(2^M x)$ . Clearly,  $u_M \in L^2(\mathcal{G}_0, m)$ . Also, for  $u \in L^2(\mathcal{G}_0, m)$  let  $u_{-M}(x) = 2^{-Md/2} u(2^{-M} x)$ ,  $x \in \mathcal{G}_M$ .

We will need the following scaling properties of Dirichlet forms and principal eigenvalues.

**Lemma 4.3** *Let  $S$  be a complete subordinator with Laplace exponent  $\phi$  given by (4.4). Then the following hold.*

(a) *If (U1) is satisfied, then for every  $M \in \mathbb{Z}_+$  and a potential  $0 \leq V \in \mathcal{K}^{X^M}$  we have*

$$\mathcal{E}_{\phi, V}^M(u, u) = 2^{-Md_w} b \mathcal{E}_{(d_w), \tilde{V}}^0(u_M, u_M), \quad u \in \mathcal{F}_{\phi, V}^M,$$

*and  $\mathcal{F}_{(d_w), \tilde{V}}^0 = \left\{ u \in L^2(\mathcal{G}_0, m) : u_{-M} \in \mathcal{F}_{\phi, V}^M \right\}$  with  $\tilde{V}(x) := \frac{2^{Md_w}}{b} V(2^M x)$ ,  $x \in \mathcal{G}_0$ . In particular,*

$$\lambda_1^M(\phi, V) = 2^{-Md_w} b \lambda_1^0(d_w, \tilde{V}).$$

(b) If **(U2)** or **(U3)** is satisfied, then there is a constant  $c_{4.4} = c_{4.4}(\phi) \in (0, 1]$  such that for every  $M \in \mathbb{Z}_+$  and a potential  $0 \leq V \in \mathcal{K}^{X^M}$ , we have

$$\mathcal{E}_{\phi, V}^M(u, u) \geq c_{4.4} 2^{-M\alpha_1} \mathcal{E}_{(\alpha_1), \tilde{V}}^0(u_M, u_M), \quad u \in \mathcal{F}_{\phi, V}^M,$$

and  $\mathcal{F}_{(\alpha_1), \tilde{V}}^0 \supseteq \left\{ u \in L^2(\mathcal{G}_0, m) : u_{-M} \in \mathcal{F}_{\phi, V}^M \right\}$  with  $\tilde{V}(x) := \frac{2^{M\alpha_1}}{c_{4.4}} V(2^M x)$ ,  $x \in \mathcal{G}_0$ . In particular,

$$\lambda_1^M(\phi, V) \geq 2^{-M\alpha_1} c_{4.4} \lambda_1^0(\alpha_1, \tilde{V}).$$

**Proof.** We only prove (b). The assertion (a) follows directly by definitions of Dirichlet forms and exactly the same arguments.

Assume first that **(U3)** holds with some  $\alpha_1 = \alpha_2 \in (0, d_w)$ . Let  $M \in \mathbb{Z}_+$  and  $0 \leq V \in \mathcal{K}^{X^M}$ . In this case we have  $\mathcal{F}_{\phi, V}^M = \mathcal{F}_{\phi}^M \cap L^2(\mathcal{G}_M, V(x)m(dx))$ . Since  $b = 0$ , for every  $u \in \mathcal{F}_{\phi, V}^M$ , we have, with  $J^M$  given by (4.13),

$$\mathcal{E}_{\phi, V}^M(u, u) = \int_{\mathcal{G}_M \times \mathcal{G}_M} (u(x) - u(y))^2 J_{\phi}^M(x, y) m(dx) m(dy) + \int_{\mathcal{G}_M} V(x) u^2(x) m(dx).$$

By Lemma 4.1, we have  $\rho(s) \geq c_{4.1} s^{-1-\alpha_1/d_w}$ ,  $s > 0$ . Therefore for every  $u \in \mathcal{F}_{\phi, V}^M$  we get

$$\int_{\mathcal{G}_M \times \mathcal{G}_M} (u(x) - u(y))^2 J_{\phi}^M(x, y) m(dx) m(dy) \geq c c_{4.1} \mathcal{E}_{(\alpha_1)}^M(u, u), \quad \text{with } c = c(d, \alpha_1),$$

and, consequently,

$$\mathcal{E}_{\phi, V}^M(u, u) \geq c c_{4.1} \mathcal{E}_{(\alpha_1)}^M(u, u) + \int_{\mathcal{G}_M} V(x) u^2(x) m(dx). \quad (4.14)$$

We now show that under **(U2)** the inequality as in (4.14) also holds, but an extra step is needed. Let  $u \in \mathcal{F}_{\phi, V}^M = \mathcal{F}_{(d_w), V}^M$ . Using the estimates from Lemma 4.1 and Fubini we will found the lower bound on  $I_M(u) := \int_{\mathcal{G}_M \times \mathcal{G}_M} (u(x) - u(y))^2 J_{\phi}^M(x, y) m(dx) m(dy)$ . We can write, with any  $\delta < 1$ :

$$\begin{aligned} I_M(u) &\geq c_{4.1} (\delta/2) \int_{\mathcal{G}_M \times \mathcal{G}_M} \int_1^\infty (u(x) - u(y))^2 s^{-1-\alpha_1/d_w} g^M(s, x, y) ds m(dx) m(dy) \\ &= c_{4.1} (\delta/2) \left( \int_{\mathcal{G}_M \times \mathcal{G}_M} \int_0^\infty (u(x) - u(y))^2 s^{-1-\alpha_1/d_w} g^M(s, x, y) ds m(dx) m(dy) \right. \\ &\quad \left. - \int_{\mathcal{G}_M \times \mathcal{G}_M} \int_0^1 (u(x) - u(y))^2 s^{-1-\alpha_1/d_w} g^M(s, x, y) ds m(dx) m(dy) \right). \end{aligned}$$

In the first of the integrals in the last formula we recognize (up to a constant) the Dirichlet form of the process  $X_{(\alpha_1)}^M$ , while the other integral is an error term (denoted by  $E_M(u)$ ) which we will now estimate. Using Fubini again we write:

$$E_M(u) = c_{4.1} \delta \int_0^1 \left( \frac{1}{2s} \int_{\mathcal{G}_M \times \mathcal{G}_M} (u(x) - u(y))^2 g^M(s, x, y) m(dx) m(dy) \right) s^{-\alpha_1/d_w} ds. \quad (4.15)$$

For any  $s > 0$  we have

$$\frac{1}{2s} \int_{\mathcal{G}_M \times \mathcal{G}_M} (u(x) - u(y))^2 g^M(s, x, y) m(dx) m(dy) \leq \mathcal{E}_{(d_w)}^M(u, u)$$

(this is so because the approximating forms increase towards  $\mathcal{E}_{(d_w)}^M(u, u)$  as  $s \downarrow 0$ ). Inserting this bound inside (4.15) and integrating from 0 to 1 we end up with the estimate

$$E_M(u) \leq (c_{4.1}\delta d_w)/(d_w - \alpha_1) \mathcal{E}_{(d_w)}^M(u, u).$$

For  $\delta = (d_w - \alpha_1)/d_w ((b/c_{4.1}) \wedge 1)$ , we have  $E_M(u) \leq (b \wedge c_{4.1}) \mathcal{E}_{(d_w)}^M(u, u)$ , therefore we get

$$\begin{aligned} \mathcal{E}_{\phi, V}^M(u, u) &= b \mathcal{E}_{(d_w)}^M(u, u) + \int_{\mathcal{G}_M \times \mathcal{G}_M} (u(x) - u(y))^2 J_\phi^M(x, y) m(dx) m(dy) + \int_{\mathcal{G}_M} V(x) u^2(x) m(dx) \\ &\geq b \mathcal{E}_{(d_w)}^M(u, u) - E_M(u) + cc_{4.1}(\delta/2) \mathcal{E}_{(\alpha_1)}^M(u, u) + \int_{\mathcal{G}_M} V(x) u^2(x) m(dx) \\ &\geq cc_{4.1}(\delta/2) \mathcal{E}_{(\alpha_1)}^M(u, u) + \int_{\mathcal{G}_M} V(x) u^2(x) m(dx). \end{aligned}$$

This is exactly (4.14), with a smaller constant  $c_{4.4} = cc_{4.1}(\delta/2)$ . In the sequel, we just write  $c_{4.4}$  in either case.

Next, one can directly check using (4.9) and (2.5) that  $\mathcal{E}_{(\alpha_1)}^M(u, u) = 2^{-M\alpha_1} \mathcal{E}_{(\alpha_1)}^0(u_M, u_M)$ . This way we obtain

$$\mathcal{E}_{\phi, V}^M(u, u) \geq 2^{-M\alpha_1} c_{4.4} \left( \mathcal{E}_{(\alpha_1)}^0(u_M, u_M) + \int_{\mathcal{G}_0} \tilde{V}(x) u_M^2(x) m(dx) \right) = 2^{-M\alpha_1} c_{4.4} \mathcal{E}_{(\alpha_1), \tilde{V}}^0(u_M, u_M),$$

with  $\tilde{V}(x) := (2^{M\alpha_1}/c_{4.4})V(2^M x)$ ,  $x \in \mathcal{G}_0$ . This inequality also implies that

$$\mathcal{F}_{(\alpha_1), \tilde{V}}^0 \supseteq \{u \in L^2(\mathcal{G}_0, m) : u_{-M} \in \mathcal{F}_{\phi, V}^M\}.$$

To prove the inequality between principal eigenvalues it suffices to use the standard variational formulas for eigenvalues:

$$\lambda_1^M(\phi, V) = \inf_{u \in \mathcal{F}_{\phi, V}^M} \frac{\mathcal{E}_{\phi, V}^M(u, u)}{\|u\|_{L^2(\mathcal{G}_M, m)}^2} \quad \text{and} \quad \lambda_1^0(\alpha_1, \tilde{V}) = \inf_{u \in \mathcal{F}_{(\alpha_1), \tilde{V}}^0} \frac{\mathcal{E}_{(\alpha_1), \tilde{V}}^0(u, u)}{\|u\|_{L^2(\mathcal{G}_0, m)}^2}.$$

Indeed, by the arguments above, for  $u \in \mathcal{F}_{\phi, V}^M$  we have  $u_M \in \mathcal{F}_{(\alpha_1), \tilde{V}}^0$  and  $\mathcal{E}_{\phi, V}^M(u, u) \geq c_{4.4} 2^{-M\alpha_1} \mathcal{E}_{(\alpha_1), \tilde{V}}^0(u_M, u_M)$ . Since also  $\|u\|_{L^2(\mathcal{G}_M, m)} = \|u_M\|_{L^2(\mathcal{G}_0, m)}$ , for every  $u \in \mathcal{F}_{\phi, V}^M$ , we have

$$\frac{\mathcal{E}_{\phi, V}^M(u, u)}{\|u\|_{L^2(\mathcal{G}_M, m)}^2} \geq c_{4.4} 2^{-M\alpha_1} \frac{\mathcal{E}_{(\alpha_1), \tilde{V}}^0(u_M, u_M)}{\|u_M\|_{L^2(\mathcal{G}_0, m)}^2} \geq c_{4.4} 2^{-M\alpha_1} \inf_{v \in \mathcal{F}_{(\alpha_1), \tilde{V}}^0} \frac{\mathcal{E}_{(\alpha_1), \tilde{V}}^0(v, v)}{\|v\|_{L^2(\mathcal{G}_0, m)}^2} = c_{4.4} 2^{-M\alpha_1} \lambda_1^0(\alpha_1, \tilde{V}).$$

By taking the infimum on the left hand side over all functions  $u \in \mathcal{F}_{\phi, V}^M$ , we get the desired inequality between the principal eigenvalues. The proof is complete.  $\square$

#### 4.2.2 Random Feynman-Kac semigroup and periodization of the Poissonian potential

Recall that  $V$  is called a random Poissonian potential on  $\mathcal{G}$  if it is given by (2.8). Below we study the process  $X^M$  perturbed by the Poissonian potentials  $V(x, \omega)$ ,  $x \in \mathcal{G}_M$ ,  $\omega \in \Omega$ , with profiles  $W$  satisfying all conditions **(W1)**–**(W3)** and restricted to  $\{(x, y) : x, y \in \mathcal{G}_M\}$ . As proved in [11, Proposition 2.1], under the condition **(W1)** we have  $V(\cdot, \omega) \in \mathcal{K}_{loc}^X$  and  $V(\cdot, \omega) \in \mathcal{K}^{X^M}$ ,  $\mathbb{Q}$ -almost surely. The corresponding Feynman-Kac semigroup will be denoted by  $(T_t^{\phi, V, M, \omega})_{t \geq 0}$  (resp.  $(T_t^{\gamma, V, M, \omega})_{t \geq 0}$  for  $X_{(\gamma)}^M$  with  $\gamma \in (0, d_w]$ ). For every  $t > 0$ , the eigenvalues of operators  $T_t^{\phi, V, M, \omega}$  are given by

$\{\exp(-t\lambda_n^M(\phi, V, \omega))\}_{n \geq 1}$  (resp.  $\{\exp(-t\lambda_n^M(\gamma, V, \omega))\}_{n \geq 1}$ ), where the random variables  $\lambda_n^M(\phi, V, \omega)$  can be ordered as  $0 \leq \lambda_1^M(\phi, V, \omega) < \lambda_2^M(\phi, V, \omega) \leq \lambda_3^M(\phi, V, \omega) \leq \dots \rightarrow \infty$ , for  $\mathbb{Q}$ -almost all  $\omega$ .

Our further argument uses some special 'periodization' of the Poissonian potential  $V$ , introduced recently in [11, Def. 3.1]: the family of random fields  $(V_M^*)_{M \in \mathbb{Z}_+}$  on  $\mathcal{G}$  given by

$$V_M^*(x, \omega) := \int_{\mathcal{G}_M} \sum_{y' \in \pi_M^{-1}(y)} W(x, y') \mu^\omega(dy), \quad M \in \mathbb{Z}_+, \quad (4.16)$$

is called the  $M$ -periodization of  $V$  in the Sznitman sense. The same argument as in [11, Proposition 2.1] yields that under **(W1)**, for  $\mathbb{Q}$ -almost all  $\omega \in \Omega$ , one has  $V_M^*(\cdot, \omega) \in \mathcal{K}^{X^M}$ , for every  $M \in \mathbb{Z}$ .

For  $t > 0$  and  $M \in \mathbb{Z}_+$  we define:

$$L_M^{N^*}(t, \omega) = \frac{1}{m(\mathcal{G}_M)} \int_{\mathcal{G}_M} p^M(t, x, x) \mathbf{E}_{x,x}^{M,t} \left[ e^{-\int_0^t V_M^*(X_s^M, \omega) ds} \right] m(dx). \quad (4.17)$$

Our argument in this section essentially relies on the following monotonicity properties.

**Lemma 4.4** *If one of the assumptions (U1)-(U3) and all of the assumptions (W1)-(W3) hold, then for any given  $t > 0$  we have*

$$\mathbb{E}_{\mathbb{Q}} L_M^{N^*}(t, \omega) \searrow L(t) \quad \text{as } M \rightarrow \infty. \quad (4.18)$$

In particular,

$$L(t) \leq \mathbb{E}_{\mathbb{Q}} L_M^{N^*}(t, \omega), \quad M \in \mathbb{Z}_+, \quad t > 0. \quad (4.19)$$

**Proof.** [11, Proof of Theorem 3.1] □

In the sequel we will be mainly working with the following type of rescaled potentials. For a profile  $W : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}_+$ , a random measure  $\mu^\omega$  with intensity  $\nu > 0$ , and a number  $\gamma > 0$  we denote

$$V_{0,M,\gamma}^*(x, \omega) := \int_{\mathcal{G}_0} \sum_{y' \in \pi_0^{-1}(y)} 2^{M\gamma} W(2^M x, 2^M y') \mu^{M,\omega}(dy), \quad x \in \mathcal{G}_0, \quad M \in \mathbb{Z}_+, \quad (4.20)$$

where  $\mu^{M,\omega}$  is the random measure corresponding to the Poisson point process with intensity  $2^{Md}\nu$ . Clearly,  $V_{0,M,\gamma}^*$  is the 0-periodization in the Sznitman sense of the Poissonian potential, which is based on the rescaled profile  $2^{M\gamma} W(2^M x, 2^M y)$  and the random measure  $\mu^{M,\omega}$  with rescaled intensity.

### 4.2.3 Derivation of the upper bound for the short range interaction

The following upper bound will be the key point for our investigations in this subsection.

**Lemma 4.5** *Let  $S$  be a complete subordinator with Laplace exponent  $\phi$  given by (4.4) and let  $V$  be a Poissonian potential with profile  $W$  satisfying the assumptions (W1)-(W3). The following hold.*

- (a) *Under the assumption (U1), there exists a constant  $c_{4.5} > 0$  such that for every  $t > 1$  and every number  $M \in \mathbb{Z}_+$  such that  $M \leq \frac{\log_2(t/\nu)}{d+d_w}$  we have*

$$\mathbb{E}_{\mathbb{Q}}[L_M^{N^*}(t, \omega)] \leq c_{4.5} \mathbb{E}_{\mathbb{Q}} \exp \left[ -b \left( 1 - \frac{1}{t} \right) \nu^{\frac{d_w}{d+d_w}} t^{\frac{d}{d+d_w}} \lambda_1^0(d_w, V_{0,M,d_w}^*, \omega) \right], \quad (4.21)$$

where the potential  $V_{0,M,d_w}^*$  is given by (4.20).



(b) Under the assumption **(U2)** or **(U3)**, there exists a constant  $c_{4.6} > 0$  such that for every  $t > 1$  and every number  $M \in \mathbb{Z}_+$  such that  $M \leq \frac{\log_2(t/\nu)}{d+\alpha_1}$  we have

$$\mathbb{E}_{\mathbb{Q}} L_M^{N^*}(t, \omega) \leq c_{4.6} \mathbb{E}_{\mathbb{Q}} \exp \left[ -c_{4.4} \left( 1 - \frac{1}{t} \right) \nu^{\frac{\alpha_1}{d+\alpha_1}} t^{\frac{d}{d+\alpha_1}} \lambda_1^0(\alpha_1, V_{0,M,\alpha_1}^*, \omega) \right], \quad (4.22)$$

where the potential  $V_{0,M,\alpha_1}^*$  is given by (4.20).

**Proof.** We only prove (b). The proof of (a) requires exactly the same argument and is even easier. Let  $\phi$  satisfy **(U2)** or **(U3)** and let  $V$  be a Poissonian potential with profile  $W$  as in the assumptions. Fix arbitrary  $t > 1$  and  $M \in \mathbb{Z}_+$  such that  $M \leq \frac{\log_2(t/\nu)}{d+\alpha_1}$ . By Fubini, for all such  $t$  and  $M$ , we get

$$\mathbb{E}_{\mathbb{Q}} L_M^{N^*}(t, \omega) = \frac{1}{m(\mathcal{G}_M)} \int_{\mathcal{G}_M} p^M(t, x, x) \mathbf{E}_{x,x}^{M,t} \left[ \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_0^t V_M^*(X_s^M, \omega) ds} \right] \right] m(dx). \quad (4.23)$$

Observe now that by the exponential formula (2.9) and the scaling properties of the measure  $m$ , for every measurable and nonnegative function  $f$  on  $\mathcal{G}$  we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} e^{-\int_{\mathcal{G}} f(y) \mu^{\omega}(dy)} &= \exp \left( -\nu \int_{\mathcal{G}} \left( 1 - e^{-f(y)} \right) m(dy) \right) \\ &= \exp \left( -2^{Md} \nu \int_{\mathcal{G}} \left( 1 - e^{-f(2^M y)} \right) m(dy) \right) = \mathbb{E}_{\mathbb{Q}} e^{-\int_{\mathcal{G}} f^M(y) \mu^{M,\omega}(dy)}, \end{aligned}$$

where  $f^M(y) = f(2^M y)$ ,  $y \in \mathcal{G}$ , and  $\mu^{M,\omega}$  is the random measure corresponding to the Poisson point process with rescaled intensity  $2^{Md} \nu$ . Applying this observation to the functions

$$\mathcal{G} \ni y \mapsto f_w(y) := \mathbf{1}_{\mathcal{G}_M}(y) \cdot \sum_{y' \in \pi_M^{-1}(\pi_M(y))} \int_0^t W(X_s^M(w), y') ds,$$

we obtain that for every  $x \in \mathcal{G}_M$  and  $\mathbf{P}_{x,x}^{M,t}$ -almost all  $w$

$$\mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_0^t V_M^*(X_s^M, t) ds} \right] = \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_0^t V_0^{M*}(X_s^M, \omega) ds} \right], \quad (4.24)$$

where

$$V_0^{M*}(x, \omega) := \int_{\mathcal{G}_0} \sum_{y' \in \pi_M^{-1}(2^M y)} W(x, y') \mu^{M,\omega}(dy) = \int_{\mathcal{G}_0} \sum_{y' \in \pi_0^{-1}(y)} W(x, 2^M y') \mu^{M,\omega}(dy).$$

Inserting (4.24) to (4.23) and using the bridge kernel representation [11, (2.28)], we thus get

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} L_M^{N^*}(t, \omega) &= \frac{1}{m(\mathcal{G}_M)} \int_{\mathcal{G}_M} p^M(t, x, x) \mathbf{E}_{x,x}^{M,t} \left[ \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_0^t V_0^{M*}(X_s^M, \omega) ds} \right] \right] m(dx) \\ &= \frac{1}{m(\mathcal{G}_M)} \mathbb{E}_{\mathbb{Q}} \text{Tr} T_t^{\phi, V_0^{M*}, M, \omega} = \frac{1}{m(\mathcal{G}_M)} \mathbb{E}_{\mathbb{Q}} \sum_{n=1}^{\infty} e^{-t} \lambda_n^M(\phi, V_0^{M*}, \omega) \\ &= \frac{1}{m(\mathcal{G}_M)} \mathbb{E}_{\mathbb{Q}} \sum_{n=1}^{\infty} e^{-(t-1)} \lambda_n^M(\phi, V_0^{M*}, \omega) e^{-\lambda_n^M(\phi, V_0^{M*}, \omega)}. \end{aligned}$$

Since for  $\mathbb{Q}$ -almost all  $\omega \in \Omega$  we have  $0 \leq \lambda_1^M(\phi, V_0^{M*}, \omega) < \lambda_2^M(\phi, V_0^{M*}, \omega) \leq \lambda_3^M(\phi, V_0^{M*}, \omega) \leq \dots$ , it follows that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} L_M^{N^*}(t, \omega) &\leq \mathbb{E}_{\mathbb{Q}} e^{-(t-1) \lambda_1^M(\phi, V_0^{M*}, \omega)} \cdot \frac{1}{m(\mathcal{G}_M)} \text{Tr} T_1^{\phi, V_0^{M*}, M, \omega} \\ &\leq \mathbb{E}_{\mathbb{Q}} e^{-(t-1) \lambda_1^M(\phi, V_0^{M*}, \omega)} \cdot \frac{1}{m(\mathcal{G}_M)} \int_{\mathcal{G}_M} p^M(1, x, x) m(dx). \end{aligned}$$

Moreover, by Lemma 4.2 we have  $p^M(1, x, x) \leq 3c_{4.3}$ , for every  $x \in \mathcal{G}_M$  and  $M \in \mathbb{Z}_+$ . Thus, we get

$$\mathbb{E}_{\mathbb{Q}} L_M^{N^*}(t, \omega) \leq 3c_{4.3} \mathbb{E}_{\mathbb{Q}} e^{-(t-1)\lambda_1^M(\phi, V_0^{M^*}, \omega)}.$$

Since  $V_{0,M,\alpha_1}^*(x) = 2^{M\alpha_1} V_0^{M^*}(2^M x)$ ,  $x \in \mathcal{G}_0$  (recall that  $V_{0,M,\alpha_1}^*$  is given by (4.20)), we derive from Lemma 4.3 (b) the inequality

$$\lambda_1^M(\phi, V_0^{M^*}, \omega) \geq c_{4.4} 2^{-M\alpha_1} \lambda_1^0(\alpha_1, V_{0,M,\alpha_1}^*, \omega), \quad M \in \mathbb{Z}_+,$$

which holds for  $\mathbb{Q}$ -almost all  $\omega$ . In consequence,

$$\mathbb{E}_{\mathbb{Q}} L_M^{N^*}(t, \omega) \leq 3c_{4.3} \mathbb{E}_{\mathbb{Q}} e^{-c_{4.4}(t-1)2^{-M\alpha_1} \lambda_1^0(\alpha_1, V_{0,M,\alpha_1}^*, \omega)}.$$

Therefore, for every  $t > 1$  and  $M \in \mathbb{Z}_+$  such that  $2^M \leq (t/\nu)^{\frac{1}{d+\alpha_1}}$  we finally get

$$\mathbb{E}_{\mathbb{Q}} L_M^{N^*}(t, \omega) \leq 3c_{4.3} \mathbb{E}_{\mathbb{Q}} \exp \left[ -c_{4.4} \left( 1 - \frac{1}{t} \right) \nu^{\frac{\alpha_1}{d+\alpha_1}} t^{\frac{d}{d+\alpha_1}} \lambda_1^0(\alpha_1, V_{0,M,\alpha_1}^*, \omega) \right].$$

The proof is complete. □

Under the following additional assumption on the profile  $W$ :

**(W4)** There exist constants  $a_0, A > 0$  such that

$$W(x, y) \geq A \text{ when } d(x, y) \leq a_0$$

we prove the following theorem.

**Theorem 4.1** *Let  $X$  be a subordinate Brownian motion in  $\mathcal{G}$  via the subordinator  $S$  with Laplace exponent  $\phi$  of the form (4.4) and let  $V$  be a Poissonian potential with the profile  $W$  such that the assumptions **(W1)**-(**W4**) are satisfied. Then there exists  $D_1 > 0$  such that the following hold.*

(a) *Under the assumption **(U1)**:*

$$\limsup_{t \rightarrow \infty} \frac{\log L(t)}{t^{\frac{d}{d+d_w}}} \leq -D_1 \nu^{\frac{d_w}{d+d_w}}. \quad (4.25)$$

(b) *Under the assumption **(U2)** or **(U3)**:*

$$\limsup_{t \rightarrow \infty} \frac{\log L(t)}{t^{\frac{d}{d+\alpha_1}}} \leq -D_1 \nu^{\frac{\alpha_1}{d+\alpha_1}}. \quad (4.26)$$

**Proof.** In both cases (a), (b) we use Sznitman's theorem from the Appendix, in either its diffusion version [25, Theorem 1.4] or the non-diffusion version [13, Theorem 1], adapted to the potential case. As the statements of both these theorems are nearly identical (they pertain to either  $d_w$  or  $\alpha_1 \in (0, d_w)$ ), we will write ' $\gamma$ ' for  $d_w$  or  $\alpha_1 \in (0, d_w)$ , depending on the context.

Let now

$$M = M(t) = \left\lceil \frac{\log_2(t/\nu)}{(d+\gamma)} \right\rceil, \quad \text{i.e.} \quad 2^M \leq \left( \frac{t}{\nu} \right)^{1/(d+\gamma)} < 2^{M+1}, \quad (4.27)$$

and write  $\epsilon = 2^{-M}$ . By Lemmas 4.4 and 4.5, for every  $t > 1$  and  $M = M(t)$  given by (4.27) we have

$$L(t) \leq \mathbb{E}_{\mathbb{Q}} L_M^{N^*}(t, \omega) \leq c_{4.6} \mathbb{E}_{\mathbb{Q}} \exp \left[ -c_{4.4} \left( 1 - \frac{1}{t} \right) \nu^{\frac{\gamma}{d+\gamma}} t^{\frac{d}{d+\gamma}} \lambda_1^0(\gamma, V_{0,M,\gamma}^*, \omega) \right],$$

with  $b$  and  $c_{4.5}$  replacing  $c_{4.4}$  and  $c_{4.6}$  in case (a). Hence, it is enough to estimate the expectation on the right hand side of the formula above, independently of  $M$ .

Fix a number  $a > a_0$  and denote  $W_a(x, y) = W(x, y) \cdot \mathbf{1}_{d(x,y) \leq a}$ ,  $x, y \in \mathcal{G}$ . The number  $a$  will not vary throughout the proof. The periodized potential  $V_{0,M,\gamma}^*(x)$ ,  $x \in \mathcal{G}_0$ , satisfies

$$V_{0,M,\gamma}^*(x, \omega) \geq \int_{\mathcal{G}_0} 2^{M\gamma} W_a(2^M x, 2^M y) \mu^{M,\omega}(dy) =: \tilde{V}_{0,M,\gamma}^*(x, \omega),$$

where now  $\mu^{M,\omega}$  comes from the rescaled cloud on  $\mathcal{G}_0$  with intensity  $\tilde{\nu} = 2^{Md}\nu$ , whose law will still be denoted by  $\mathbb{Q}$ . The new profile  $\mathcal{G}_0 \times \mathcal{G}_0 \ni (x, y) \mapsto W_{a,M}(x, y) = 2^{M\gamma} W_a(2^M x, 2^M y)$  has range  $a2^{-M} = a\epsilon$ , and its values are bigger than  $2^{M\gamma}A$  when  $d(x, y) \leq a_0\epsilon$ .

Further assume that  $b = 2^\kappa > a$ , with  $\kappa \in \mathbb{Z}$ , and let  $K, \delta > 0$  be given.

$\mathbb{Q}$ -almost surely, there is a finite number of Poisson points in  $\mathcal{G}_0$ . From now on we will be working with a fixed configuration  $\omega = (x_1, \dots, x_N) \subset \mathcal{G}_0$  of Poisson points. We divide them into ‘good’ and ‘bad’ points according to Definition 7.1, and remove the closed balls  $B(x_i, b\epsilon)$  with centers at good points from the state-space. We are left with the set

$$\Theta_{b,M} = \mathcal{G}_0 \setminus \bigcup_{x_i - \text{good}} \overline{B}(x_i, b\epsilon),$$

and we let the process  $X_{(\gamma)}^0$  evolve in this set, being killed when it enters one of the balls  $\overline{B}(x_i, b\epsilon)$ ,  $x_i - \text{good}$ . Let  $\lambda_1^0(\gamma, \Theta_{b,M}, \omega)$  be the principal eigenvalue of the generator of this process.

The assumptions of Theorem 7.1 are fulfilled (we postpone their verification until after the proof; see Subsection 4.3), and so there exists  $\epsilon_0 > 0$ , depending on the process, the potential  $W$ , and the numbers  $a, b, K, \delta, \gamma$  (not on  $M$ ) such that when  $\epsilon < \epsilon_0$ , then

$$\begin{aligned} \lambda_1^0(\gamma, \Theta_{b,M}, \omega) \wedge K &\leq \lambda_1^0(\gamma, \tilde{V}_{0,M,\gamma}^*, \omega) \wedge K + \delta \\ &\leq \lambda_1^0(\gamma, V_{0,M,\gamma}^*, \omega) \wedge K + \delta \end{aligned} \quad (4.28)$$

(the last inequality follows from the inequality  $V_{0,M,\gamma}^* \geq \tilde{V}_{0,M,\gamma}^*$ , combined with the variational definition of the principal eigenvalue). In particular, since  $\epsilon = 2^{-M}$ , there exists  $M_0$  such that for  $M > M_0$  the relation (4.28) holds. The way  $M = M(t)$  was defined (see (4.27)) gives that there exists  $t_0 \geq 1$  such that it holds for  $t > t_0$ .

The conclusion of the proof is much alike the conclusion of [25, Theorem 1.7] or [19, Lemma 9]. Let  $M > M_0$  (equivalently:  $t > t_0$ ). We chop the sides of the triangle  $\mathcal{G}_0$  into  $(b\epsilon)^{-1} = 2^{M-\kappa}$  parts, which yields  $N(b, M) = 2^{(M-\kappa)d} = (b\epsilon)^{-d}$  small gasket triangles of sidelength  $b\epsilon$ . Now, instead of removing balls  $\overline{B}(x_i, b\epsilon)$  from the state-space, we remove those closed small triangles that received some (good) Poisson points. More precisely, let  $A_{b,M}$  be the union of those small triangles that received some Poisson points, and  $\hat{A}_{b,M}$  – of those triangles that received some good Poisson points. We set

$$U_{b,M} = \mathcal{G}_0 \setminus A_{b,M}, \quad \hat{U}_{b,M} = \mathcal{G}_0 \setminus \hat{A}_{b,M}.$$

As the diameter of each of the triangles removed equals to  $b\epsilon$ , we have  $\Theta_{b,M} \subset \hat{U}_{b,M}$ , and consequently  $\lambda_1^0(\gamma, \Theta_{b,M}, \omega) \geq \lambda_1^0(\gamma, \hat{U}_{b,M}, \omega)$ , where  $\lambda_1^0(\gamma, \hat{U}_{b,M}, \omega)$  is the principal eigenvalue of the process that is killed upon exiting  $\hat{U}_{b,M}$ .

Altogether, for any given configuration  $\omega$ , given  $K, \delta, b$ , we have

$$\begin{aligned}\lambda_1^0(\gamma, V_{0,M,\gamma}^*, \omega) &\geq \lambda_1^0(\gamma, V_{0,M,\gamma}^*, \omega) \wedge K \geq \lambda_1^0(\gamma, \Theta_{b,M}, \omega) \wedge K - \delta \\ &\geq \lambda_1^0(\gamma, \widehat{U}_{b,M}, \omega) \wedge K - \delta.\end{aligned}$$

Denoting by  $\mathcal{U}_{b,M}$  the collection of all possible configurations of the sets  $U_{b,M}$  and  $\widehat{U}_{b,M}$  and noting that  $\#\mathcal{U}_{b,M} = 2^{N(b,M)}$ , we can proceed as follows (taking the precisely chosen  $M = M(t)$  and  $t > t_0$ ):

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}} \exp \left[ -c_{4.4} \left( 1 - \frac{1}{t} \right) \nu^{\frac{\gamma}{d+\gamma}} t^{\frac{d}{d+\gamma}} \lambda_1^0(\gamma, V_{0,M,\gamma}^*, \omega) \right] \\ &\leq \mathbb{E}_{\mathbb{Q}} \exp \left[ -c_{4.4} \left( 1 - \frac{1}{t} \right) \nu^{\frac{\gamma}{d+\gamma}} t^{\frac{d}{d+\gamma}} (\lambda_1^0(\gamma, \widehat{U}_{b,M}, \omega) \wedge K - \delta) \right] \\ &\leq \sum_{U, \widehat{U} \in \mathcal{U}_{b,M}} \mathbb{E}_{\mathbb{Q}} \left[ \exp \left( -c_{4.4} \left( 1 - \frac{1}{t} \right) \nu^{\frac{\gamma}{d+\gamma}} t^{\frac{d}{d+\gamma}} (\lambda_1^0(\gamma, \widehat{U}_{b,M}, \omega) \wedge K - \delta) \right) \mathbf{1}_{\{U_{b,M} = U, \widehat{U}_{b,M} = \widehat{U}\}} \right].\end{aligned}$$

For any  $A \in \mathcal{B}(\mathcal{G}_0)$  we have  $\mathbb{Q}[\mathcal{N}(A) = 0] = e^{-\tilde{\nu}m(A)}$ . One knows [25, Lemma 1.3] that  $m(\bigcup_{x_i \text{--bad}} \overline{B}(x_i, b\epsilon)) \leq \delta$ , therefore also  $m(A_{b,M} \setminus \widehat{A}_{b,M}) \leq \delta$  and  $m(\widehat{U}_{b,M}) \leq m(U_{b,M}) + \delta$ . Therefore the estimate continues as

$$= \sum_{(U, \widehat{U}) \in \mathcal{A}_{b,M}} \exp \left( -c_{4.4} \left( 1 - \frac{1}{t} \right) \nu^{\frac{\gamma}{d+\gamma}} t^{\frac{d}{d+\gamma}} (\lambda_1^0(\gamma, \widehat{U}) \wedge K - \delta) \right) \mathbb{Q}[U_{b,M} = U, \widehat{U}_{b,M} = \widehat{U}] =: I,$$

where  $\mathcal{A}_{b,M} \subset \mathcal{U}_{b,M} \times \mathcal{U}_{b,M}$  consists of those pairs  $(U, \widehat{U})$  for which  $U \subset \widehat{U}$ ,  $m(\widehat{U}) \leq m(U) + \delta$ , and  $\lambda_1^0(\gamma, \widehat{U})$  is the principal eigenvalue of the generator of the process  $X_{(\gamma)}^0$  killed upon exiting the open set  $\widehat{U}$ . In particular, for  $(U, \widehat{U}) \in \mathcal{A}_{b,M}$  one has

$$\mathbb{Q}[U_{b,M} = U, \widehat{U}_{b,M} = \widehat{U}] \leq e^{-\tilde{\nu}(m(\widehat{U}) - \delta)}.$$

Since

$$\#\mathcal{U}_{b,M} = 2^{N(b,M)} \leq 2^{b^{-d} \left( \frac{t}{\nu} \right)^{d/(d+\gamma)}}$$

and  $2^{-d} \nu^{\frac{\gamma}{d+\gamma}} t^{\frac{d}{d+\gamma}} \leq \tilde{\nu} = 2^{Md} \nu \leq \nu^{\frac{\gamma}{d+\gamma}} t^{\frac{d}{d+\gamma}}$ , we get

$$\begin{aligned}I &\leq \sum_{U, \widehat{U} \in \mathcal{U}_{b,M}} \exp \left( - \left[ c_{4.4} \left( 1 - \frac{1}{t} \right) \nu^{\frac{\gamma}{d+\gamma}} t^{\frac{d}{d+\gamma}} (\lambda_1^0(\gamma, \widehat{U}) \wedge K - \delta) \right] - \tilde{\nu}(m(\widehat{U}) - \delta) \right) \\ &\leq (2^{\#\mathcal{U}_{b,M}})^2 \exp \left( -c_{4.4} \left( 1 - \frac{1}{t} \right) \nu^{\frac{\gamma}{d+\gamma}} t^{\frac{d}{d+\gamma}} \inf_{U \in \mathcal{U}_{b,M}} [(\lambda_1^0(\gamma, U) \wedge K - \delta) + \frac{2^{-d}t}{c_{4.4}(t-1)}(m(U) - 2^d\delta)] \right) \\ &\leq \exp \left[ \frac{2 \log 2}{b^d} \left( \frac{t}{\nu} \right)^{\frac{d}{d+\gamma}} - c_{4.4} \left( 1 - \frac{1}{t} \right) \nu^{\frac{\gamma}{d+\gamma}} t^{\frac{d}{d+\gamma}} \inf_{U \in \mathcal{U}_0} [(\lambda_1^0(\gamma, U) \wedge K - \delta) + \frac{2^{-d}t}{c_{4.4}(t-1)}(m(U) - 2^d\delta)] \right],\end{aligned}$$

where by  $\mathcal{U}_0$  we have denoted the collection of all open subsets of  $\mathcal{G}_0$ . This bound is valid for  $t > t_0$ . In particular,

$$\begin{aligned}\limsup_{t \rightarrow \infty} \frac{\log L(t)}{t^{\frac{d}{d+\gamma}}} &\leq \frac{2 \ln 2}{b^d \nu^{\frac{d}{d+\gamma}}} - c_{4.4} \nu^{\frac{\gamma}{d+\gamma}} \inf_{U \in \mathcal{U}_0} [(\lambda_1^0(\gamma, U) \wedge K - \delta) + (2^d c_{4.4})^{-1} (m(U) - 2^d \delta)] \\ &\leq \frac{2 \ln 2}{b^d \nu^{\frac{d}{d+\gamma}}} - c_{4.4} \nu^{\frac{\gamma}{d+\gamma}} \left( \inf_{U \in \mathcal{U}_0} [\lambda_1^0(\gamma, U) + (2^d c_{4.4})^{-1} m(U)] \wedge K - \delta(1 + c_{4.4}^{-1}) \right).\end{aligned}$$

The left-hand side does not depend on  $b, K, \delta$  – by passing to the limit  $b \rightarrow \infty, \delta \rightarrow 0, K \rightarrow \infty$  on the right-hand side we get

$$\limsup_{t \rightarrow \infty} \frac{\log L(t)}{t^{\frac{d}{d+\gamma}}} \leq -c_{4.4} \nu^{\frac{\gamma}{d+\gamma}} \inf_{U \in \mathcal{U}_0} [\lambda_1^0(\gamma, U) + (2^d c_{4.4})^{-1} m(U)] =: -D_1 \nu^{\frac{\gamma}{d+\gamma}}.$$

To conclude, we need to check that  $D_1 > 0$ . We can write, for any open  $U \subset \mathcal{G}_0$ ,

$$e^{-t\lambda_1^0(\gamma, U)} \leq \text{Tr } P_t^{\gamma, 0, U} \leq \int_U p^{\gamma, 0}(t, x, x) dm(x)$$

where  $p^{\gamma, 0}$  is the transition density of the process  $X_{(\gamma)}^0$  on  $\mathcal{G}_0$ , and  $P_t^{\gamma, 0, U}$  is the semigroup of this process killed upon exiting  $U$ . In particular, for  $t > 1$ , from (4.12) with  $\alpha_1 = \alpha_2 = \beta = \gamma$  we get  $\sup_{x \in \mathcal{G}_0} p^{\gamma, 0}(t, x, x) \leq 3c_{4.3}$ , so that

$$\lambda_1^0(\gamma, U) + (2^d c_{4.4})^{-1} m(U) \geq -\frac{\log(3c_{4.3}m(U))}{t} + (2^d c_{4.4})^{-1} m(U).$$

As for  $t > 2^d c_{4.4}$  one has

$$\inf_{x \in (0,1)} \left[ -\frac{\log(3c_{4.3}x)}{t} + (2^d c_{4.4})^{-1} x \right] = \frac{1}{t} \left( 1 - \log \frac{2^d c_{4.4} \cdot 3c_{4.3}}{t} \right),$$

picking any  $t > 2^d c_{4.4} \max(1, 3c_{4.3}e^{-1})$  we achieve the desired statement that  $D_1 > 0$ .  $\square$

We also prove the matching bounds for the Feynman-Kac functionals.

**Theorem 4.2** *Let  $X$  be a subordinate Brownian motion in  $\mathcal{G}$  via the subordinator  $S$  with Laplace exponent  $\phi$  of the form (4.4) and let  $V$  be a Poissonian potential with the profile  $W$  such that the assumptions (W1)-(W4) are satisfied. Then the following hold (with  $D_1$  same as above).*

(a) *Under the assumption (U1) for any  $x \in \mathcal{G}$*

$$\limsup_{t \rightarrow \infty} \frac{\log \mathbb{E}_{\mathbb{Q}} \mathbf{E}_x [e^{-\int_0^t V(X_s, \omega) ds}]}{t^{\frac{d}{d+d_w}}} \leq -D_1 \nu^{\frac{d_w}{d+d_w}}. \quad (4.29)$$

(b) *Under the assumption (U2) or (U3) for any  $x \in \mathcal{G}$*

$$\limsup_{t \rightarrow \infty} \frac{\log \mathbb{E}_{\mathbb{Q}} \mathbf{E}_x [e^{-\int_0^t V(X_s, \omega) ds}]}{t^{\frac{d}{d+\alpha}}} \leq -D_1 \nu^{\frac{\alpha_1}{d+\alpha_1}}. \quad (4.30)$$

**Proof.** By the same argument as in [11, Ineq. (3.8)], the  $M$ -periodicity of the potential  $V_M^*$  and the definition of the measure  $\mathbf{P}^M$ , for  $M$  so large that  $x \in \mathcal{G}_M$  and for  $t > 1$ , we get

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \mathbf{E}_x \left[ e^{-\int_0^t V(X_s, \omega) ds} \right] &\leq \mathbb{E}_{\mathbb{Q}} \mathbf{E}_x \left[ e^{-\int_0^t V_M^*(X_s, \omega) ds} \right] = \mathbb{E}_{\mathbb{Q}} \mathbf{E}_x^M \left[ e^{-\int_0^t V_M^*(X_s^M, \omega) ds} \right] \\ &= \int_{\mathcal{G}_M \times \mathcal{G}_M} u^{M, \omega}(1, x, z) u^{M, \omega}(t-1, z, y) dm(z) dm(y), \end{aligned}$$

where

$$u^{M, \omega}(t, x, y) = p^M(t, x, y) \mathbf{E}_{x, y}^{M, t} \left[ e^{-\int_0^t V_M^*(X_s^M, \omega) ds} \right]$$

is the kernel of the operator  $T_t^{\phi, V_M^*, M, \omega}$  (see [11, (2.28)]). From Lemma 4.2 we have  $u^{M, \omega}(1, x, z) \leq p^M(1, x, z) \leq 3c_{4.3}$ , for every  $x, z \in \mathcal{G}_M$ . Thus, by the same estimate for the kernel  $u^{M, \omega}(t-1, z, y)$  as in [25, the last two lines on p. 235], we may conclude that

$$\mathbb{E}_{\mathbb{Q}} \mathbf{E}_x \left[ e^{-\int_0^t V(X_s, \omega) ds} \right] \leq 3c_{4.3} m(\mathcal{G}_M) \mathbb{E}_{\mathbb{Q}} \int_{\mathcal{G}_M} p^M(t-1, x, x) \mathbf{E}_{x,x}^{M, t-1} \left[ e^{-\int_0^{t-1} V_M^*(X_s^M, \omega) ds} \right] dm(x).$$

In the right-hand side we recognize the expression  $3c_{4.3} [m(\mathcal{G}_M)]^2 \mathbb{E}_{\mathbb{Q}} [L_M^{N^*}(t-1, \omega)]$ , which has been already estimated in Lemma 4.5. Therefore, starting from (4.27) and proceeding exactly in the same way as in the proof of Theorem 4.1, we finally get the desired inequality with  $\gamma = d_w$  for (a) and  $\gamma = \alpha$  for (b) respectively, with the same constant  $D_1$  as before.  $\square$

### 4.3 Verification of the assumptions of Sznitman's theorem

We apply the Sznitman's theorem (Theorem 7.1 below) in the following setting:

- \*  $\mathcal{T} = \mathcal{G}_0$ , the metric  $d$  is the shortest path distance on  $\mathcal{G}_0$ ,  $m$  is the Hausdorff measure on  $\mathcal{G}_0$  in dimension  $d = \frac{\log 3}{\log 2}$  normalized to have  $m(\mathcal{G}_0) = 1$ , which is a doubling measure;
- \* the Markov process in question is  $X_{(\gamma)}^0$  on  $\mathcal{G}_0$  for  $\gamma \in (0, d_w]$  (the reflected jump stable process or the reflected Brownian motion);
- \* for  $x, y \in \mathcal{G}_0$  the potential profile is given by  $W_M(x, y) := 2^{M\gamma} W(2^M x, 2^M y)$ , where the profile  $W$  is of finite range  $a > 0$  (i.e.  $W(x, y) = 0$  when  $d(x, y) > a$ ) and satisfies **(W1)** – **(W4)**. Assume that  $a \geq a_0$ , where  $a_0$  is the constant from **(W4)**. The range of  $W_M$  is equal to  $a2^{-M}$ ; we denote  $2^{-M} = \epsilon$ , and we will also write  $W_\epsilon$  for  $W_M$ .

All the required regularity assumptions (see Subsection 7.1) except for **(P3)** were established in [19, 25] for the reflected Brownian motion and in [14] for reflected jump stable processes. These papers were concerned with processes on  $\mathcal{G}$  evolving among killing Poissonian obstacles. We now verify the remaining condition **(P3)**, which is needed in our case.

**Proposition 4.3** *Let  $\gamma \in (0, d_w]$  and let  $X^0 = X_{(\gamma)}^0$  be the reflected  $\gamma$ -stable process on  $\mathcal{G}_0$  (not excluding the case  $\gamma = d_w$ ). Assume that the potential profile  $W$  satisfies the condition **(W4)**. Then there exists constants  $c_{4.7} = c_{4.7}(a_0, A, b, \gamma) > 0$  and  $\tau_0 = \tau_0(a_0, b, \gamma) > 0$  such that for any  $x \in \mathcal{G}_0$ ,  $\epsilon = 2^{-M} > 0$ , and  $y \in \mathcal{G}_0$  with  $d(x, y) \leq b\epsilon$  one has*

$$\mathbf{E}_x^0 \left[ e^{-\int_0^{\tau_0 \epsilon^{\gamma/2}} W_\epsilon(X_s^0, y) ds} \right] \leq 1 - 2c_{4.7}, \quad \text{with } W_\epsilon(x, y) = \epsilon^{-\gamma} W(x/\epsilon, y/\epsilon).$$

Note that the constant  $c_{4.7}$  does not depend on  $\epsilon$ , which is decisive for the proof of the upper bound theorem ( $c_{4.7}$  plays the role of the constant  $c_1$  in assumptions **(P2)**-**(P3)** in Subsection 7.1).

**Proof.** Suppose  $M \geq 0$  and  $x, y \in \mathcal{G}_0$  are as in the assumptions. Reflected processes on the gaskets  $\mathcal{G}_M$  allow for discrete scaling. Namely, for  $x \in \mathcal{G}_0$ , the processes on  $\mathcal{G}_M : (2^M X_s^0)_{s \geq 0}$  under  $\mathbf{P}_x^0$  and  $(X_{2^M \gamma s}^M)_{s \geq 0}$  under  $\mathbf{P}_{2^M x}^M$  are equal in law. Denote  $\tilde{x} = 2^M x (= \epsilon^{-1} x)$  and  $\tilde{y} = 2^M y$  so that for any  $t > 0$

$$\begin{aligned} \mathbf{E}_x^0 \left[ e^{-\int_0^{t\epsilon^\gamma} W_\epsilon(X_s^0, y) ds} \right] &= \mathbf{E}_{2^{-M}\tilde{x}}^0 \left[ e^{-\int_0^{t2^{-M\gamma}} 2^{M\gamma} W(2^M X_s^0, 2^M y) ds} \right] \\ &= \mathbf{E}_{\tilde{x}}^M \left[ e^{-\int_0^{t2^{-M\gamma}} 2^{M\gamma} W(X_{2^M \gamma s}^M, \tilde{y}) ds} \right] = \mathbf{E}_{\tilde{x}}^M \left[ e^{-\int_0^t W(X_s^M, \tilde{y}) ds} \right], \end{aligned}$$



where now  $\tilde{x}, \tilde{y} \in \mathcal{G}_M$  and  $d(\tilde{x}, \tilde{y}) \leq b$ .

For  $r > 0$  and fixed  $\tilde{y} \in \mathcal{G}_M$ , denote  $T_r = \inf\{t \geq 0 : X_t^M \in B(\tilde{y}, r)\}$ . Then we can write, in the third line using the strong Markov property at the stopping time  $T_{a_0/2}$ :

$$\begin{aligned}
\mathbf{E}_{\tilde{x}}^M \left[ e^{-\int_0^t W(X_s^M, \tilde{y}) ds} \right] &= \mathbf{E}_{\tilde{x}}^M \left[ e^{-\int_0^t W(X_s^M, \tilde{y}) ds} \mathbf{1}\{T_{a_0/2} < \tfrac{t}{2}\} \right] + \mathbf{E}_{\tilde{x}}^M \left[ e^{-\int_0^t W(X_s^M, \tilde{y}) ds} \mathbf{1}\{T_{a_0/2} \geq \tfrac{t}{2}\} \right] \\
&\leq \mathbf{E}_{\tilde{x}}^M \left[ e^{-\int_0^t W(X_s^M, \tilde{y}) ds} \mathbf{1}\{T_{a_0/2} < \tfrac{t}{2}\} \right] + \mathbf{P}_{\tilde{x}}^M [T_{a_0/2} \geq \tfrac{t}{2}] \\
&= \mathbf{E}_{\tilde{x}}^M \left\{ \mathbf{1}\{T_{a_0/2} < \tfrac{t}{2}\} \mathbf{E}_{X_{T_{a_0/2}}^M}^M \left[ e^{-\int_0^{t-T_{a_0/2}} W(X_s^M, \tilde{y}) ds} \right] \right\} + \mathbf{P}_{\tilde{x}}^M [T_{a_0/2} \geq \tfrac{t}{2}] \\
&\leq \mathbf{E}_{\tilde{x}}^M \left\{ \mathbf{1}\{T_{a_0/2} < \tfrac{t}{2}\} \mathbf{E}_{X_{T_{a_0/2}}^M}^M \left[ e^{-\int_0^{t/2} W(X_s^M, \tilde{y}) ds} \right] \right\} + \mathbf{P}_{\tilde{x}}^M [T_{a_0/2} \geq \tfrac{t}{2}] \\
&\leq \mathbf{E}_{\tilde{x}}^M \left\{ \mathbf{1}\{T_{a_0/2} < \tfrac{t}{2}\} \cdot \sup_{\xi \in B(\tilde{y}, \frac{a_0}{2})} \mathbf{E}_{\xi}^M \left[ e^{-\int_0^{t/2} W(X_s^M, \tilde{y}) ds} \right] \right\} + \mathbf{P}_{\tilde{x}}^M [T_{a_0/2} \geq \tfrac{t}{2}].
\end{aligned} \tag{4.31}$$

The expected value inside the supremum in the last line can be estimated as

$$\begin{aligned}
\mathbf{E}_{\xi}^M \left[ e^{-\int_0^{t/2} W(X_s^M, \tilde{y}) ds} \right] &= \mathbf{E}_{\xi}^M \left[ e^{-\int_0^{t/2} W(X_s^M, \tilde{y}) ds} \mathbf{1}\{\tau_{B(\tilde{y}, a_0)} \geq \tfrac{t}{2}\} \right] \\
&\quad + \mathbf{E}_{\xi}^M \left[ e^{-\int_0^{t/2} W(X_s^M, \tilde{y}) ds} \mathbf{1}\{\tau_{B(\tilde{y}, a_0)} < \tfrac{t}{2}\} \right] \\
&\leq e^{-At/2} \mathbf{P}_{\xi}^M [\tau_{B(\tilde{y}, a_0)} \geq \tfrac{t}{2}] + \mathbf{P}_{\xi}^M [\tau_{B(\tilde{y}, a_0)} < \tfrac{t}{2}],
\end{aligned}$$

where  $\tau_{B(\tilde{y}, a_0)}$  denotes the first exit time of the process from the ball  $B(\tilde{y}, a_0)$ . This estimate is in the form  $z \leq x(1 - y) + y$ , with  $x \in (0, 1)$  and  $y \in [0, 1]$ , so if we can prove that  $y \leq p \in (0, 1)$ , then we will also have  $z \leq x(1 - p) + p$ .

So far, all the estimates pertained to the projected stable process  $X_{(\gamma)}^M$ . Observe that for any open subset  $F \subset \mathcal{G}_M$ ,  $x \in \mathcal{G}_M$ , and  $s > 0$  one has  $\mathbf{P}_x^M [\tau_F < s] \leq \mathbf{P}_x [\tau_F < s]$ , so that it is enough to estimate  $\mathbf{P}_{\xi} [\tau_{B(\tilde{y}, a_0)} < \tfrac{t}{2}]$  for the nonprojected process (i.e. the  $\gamma$ -stable process on  $\mathcal{G}$ ). As in present case  $\xi \in B(\tilde{y}, a_0/2)$ , we have

$$\mathbf{P}_{\xi} [\tau_{B(\tilde{y}, a_0)} < \tfrac{t}{2}] \leq \mathbf{P}_{\xi} [\sup_{s \leq t/2} d(X_s, X_0) > \tfrac{a_0}{2}],$$

which is not bigger than  $\leq ct/a_0^\gamma$  in the jump stable ( $\gamma < d_w$ ) case [3, Lemma 4.3], and not bigger than  $ce^{-c_2(d_w)(\frac{a_0}{t^{1/d_w}})^{d_w/(d_w-1)}}$  in the Brownian motion ( $\gamma = d_w$ ) case [1, Theorem 4.3], where  $c = c(\gamma)$ . In the sequel, we continue with the jump stable case only, the other case follows identically (easier in fact). The important feature of these estimates is that they do no longer depend on  $M$ . Therefore, for any  $0 < t \leq t'_0 = \frac{1}{2} \frac{a_0^\gamma}{c}$ ,

$$\sup_{\xi \in B(\tilde{y}, a_0/2)} \mathbf{E}_{\xi}^M \left[ e^{-\int_0^{t/2} W(X_s^M, \tilde{y}) ds} \right] \leq e^{-At/2} \left( 1 - \frac{ct}{a_0^\gamma} \right) + \frac{ct}{a_0^\gamma} \leq \frac{1}{2} \left( e^{-At/2} + 1 \right).$$

Insert this estimate into (4.31) and continue in the same vein. This time, use the observation that

$$\mathbf{P}_{\tilde{x}}^M [T_{a_0/2} > \tfrac{t}{2}] \leq \mathbf{P}_{\tilde{x}}^M [X_{t/2}^M \notin B(\tilde{y}, a_0/2)] = 1 - \mathbf{P}_{\tilde{x}}^M [X_{t/2}^M \in B(\tilde{y}, a_0/2)] \leq 1 - \mathbf{P}_{\tilde{x}} [X_{t/2} \in B(\tilde{y}, a_0/2)].$$

Recall that the transition density  $p$  of the (nonreflected) jump  $\gamma$ -stable process in  $\mathcal{G}$  enjoys the estimate (see [3, Theorem 3.1] and [5, Theorem 1.1])

$$\frac{1}{c^{(1)}} \min \left( \frac{t}{d(x, y)^{d+\gamma}}, t^{-d/\gamma} \right) \leq p(t, x, y) \leq c^{(1)} \min \left( \frac{t}{d(x, y)^{d+\gamma}}, t^{-d/\gamma} \right), \tag{4.32}$$

with certain positive constant  $c^{(1)} = c^{(1)}(\gamma)$ . Moreover, since  $d(\tilde{x}, \tilde{y}) \leq b$ , for  $z \in B(\tilde{y}, a_0/2)$  one has  $d(\tilde{x}, z) \leq b + a_0/2$ . Consequently, from (4.32) we get, with  $c^{(2)} = c^{(2)}(\gamma, d)$ ,

$$\mathbf{P}_{\tilde{x}}[X_{t/2} \in B(\tilde{y}, a_0/2)] \geq m(B(\tilde{y}, a_0/2)) \cdot \inf_{z \in B(\tilde{y}, a_0/2)} p(t/2, \tilde{x}, z) \geq c^{(2)} a_0^d \min \left( \frac{t}{(b + \frac{a_0}{2})^{d+\gamma}}, t^{-d/\gamma} \right).$$

When  $t \leq t_0'' = (b + a_0/2)^\gamma$ , then this estimate is

$$\mathbf{P}_{\tilde{x}}[X_{t/2} \in B(\tilde{y}, a_0/2)] \geq \frac{c^{(2)} t a_0^d}{(b + \frac{a_0}{2})^{d+\gamma}},$$

being a number smaller than one if  $t \leq t_0''' = \frac{(b+a_0/2)^{d+\gamma}}{c^{(2)} a_0^d}$ . Collecting all of these, we obtain

$$\mathbf{E}_{\tilde{x}}^M \left[ e^{-\int_0^t W(X_s^M, \tilde{y}) ds} \right] \leq \frac{1}{2} \frac{c^{(2)} t a_0^d}{(b + \frac{a_0}{2})^{d+\gamma}} \left( e^{-At/2} + 1 \right) + \left( 1 - \frac{c^{(2)} t a_0^d}{(b + \frac{a_0}{2})^{d+\gamma}} \right).$$

For  $t^* = \min(t_0', t_0'', t_0''')$  we get an estimate in the form

$$\frac{p}{2} (e^{-At^*/2} + 1) + (1 - p) =: 1 - 2c_{4.7},$$

with  $p < 1$ . To conclude, we choose  $\tau_0 = 2t^*$ . The resulting constant  $c_{4.7}$  is strictly positive and depends on  $A, a_0, b, \gamma$  only. The proof is complete.  $\square$

#### 4.4 The upper bound for general potentials

We have the following statement, matching in general setting the lower bound from Theorem 3.1.

**Theorem 4.3** *Suppose  $X$  is a subordinate Brownian motion in  $\mathcal{G}$  via a complete subordinator  $S$  with Laplace exponent  $\phi$  of the form (2.4), satisfying (U1), (U2), or (U3). Let the profile  $W$  satisfy (W1) – (W4), and suppose that for certain  $\theta > 0$  there is a number  $K \in [0, \infty)$  such that*

$$K = \liminf_{d(x,y) \rightarrow \infty} W(x, y) d(x, y)^{d+\theta}.$$

Let  $\gamma = d_w$  (under (U1)) or  $\gamma = \alpha_1$  (under (U2) and (U3)). Then there exist constants  $E_1, E_1' > 0$  such that:

(i) when  $\gamma < \theta$  then

$$\limsup_{t \rightarrow \infty} \frac{\log L(t)}{t^{d/(d+\gamma)}} \leq -E_1 \nu^{\gamma/(d+\gamma)},$$

(ii) when  $\gamma = \theta$  then

$$\limsup_{t \rightarrow \infty} \frac{\log L(t)}{t^{d/(d+\gamma)}} \leq -E_1 \nu^{\gamma/(d+\gamma)} - E_1' \nu,$$

(iii) when  $\gamma > \theta$  then

$$\limsup_{t \rightarrow \infty} \frac{\log L(t)}{t^{d/(d+\theta)}} \leq -E_1' \nu.$$

**Proof.** The statements follows immediately from Proposition 4.1 and Theorem 4.1. Both statements are true in present setting, so we can use the arithmetic mean of both the bounds. The constants we obtain are:  $E_1 = \frac{1}{2} D_1$ ,  $E_1' = \frac{K}{2c_{2.2} e^K}$ .  $\square$

**Remark 4.1** Identical statements hold true for the Feynman-Kac functionals. We skip the proof.

## 5 The Lifschitz tail for the integrated density of states

In this section we transform the bounds from Theorems 3.1 and 4.3 into bounds concerning the rate of decay of  $l$  near zero. This is done by means of an exponential Tauberian theorem [9, Th. 2.1].

**Theorem 5.1** *Suppose that the assumptions of Theorem 3.1 are met. Then there exist constants  $\tilde{C}_1, \dots, \tilde{C}_4 > 0$  such that:*

(i) *when  $\beta < \theta$  then*

$$\liminf_{x \rightarrow 0} x^{d/\beta} \log l([0, x]) \geq -\tilde{C}_1 \nu,$$

(ii) *when  $\beta = \theta$  then*

$$\liminf_{x \rightarrow 0} x^{d/\beta} \log l([0, x]) \geq -\tilde{C}_2 \nu - \tilde{C}_3 \nu^{1+d/\beta},$$

(iii) *when  $\beta > \theta$  then*

$$\liminf_{x \rightarrow 0} x^{d/\theta} \log l([0, x]) \geq -\tilde{C}_4 \nu^{1+d/\theta}.$$

**Theorem 5.2** *Suppose that the assumptions of Theorem 4.3 are met. Then there exist constants  $\tilde{D}_1, \dots, \tilde{D}_4 > 0$  such that:*

(i) *when  $\gamma < \theta$  then*

$$\limsup_{x \rightarrow 0} x^{d/\gamma} \log l([0, x]) \leq -\tilde{D}_1 \nu,$$

(ii) *when  $\gamma = \theta$  then*

$$\limsup_{x \rightarrow 0} x^{d/\gamma} \log l([0, x]) \leq -\tilde{D}_2 \nu - \tilde{D}_3 \nu^{1+d/\gamma},$$

(iii) *when  $\gamma > \theta$  then*

$$\limsup_{x \rightarrow 0} x^{d/\theta} \log l([0, x]) \leq -\tilde{D}_4 \nu^{1+d/\theta}.$$

The most interesting case is that of  $\beta = \gamma$ , i.e. the case when the lower and upper scaling exponents for  $\phi$  coincide. In this case, the rate of decay of  $l([0, x])$  as  $x \rightarrow 0^+$  is of order  $e^{-\text{const} \cdot x^{-d/\beta}}$ . Likewise, when  $\theta < \beta$ , i.e., when the behaviour of the potential at infinity dominates the behaviour of the process, then the rate of decay of  $l([0, x])$  is  $e^{-\text{const} \cdot x^{-d/\theta}}$ .

## 6 Examples

At the very end, we give various examples of subordinators with Laplace exponents that are complete Bernstein functions satisfying the regularity assumptions needed for our work.

**Example 6.1** We first discuss some examples of functions  $\phi$  satisfying all of our assumptions.

- (1) *Pure drift.* Let  $\phi(\lambda) = b\lambda$ ,  $b > 0$ . The corresponding subordinate process is just the Brownian motion with speed  $b > 0$ . Clearly, the assumption **(U1)** is satisfied and **(L1)** holds with  $\beta = d_w$ .

The next two examples are jump subordinators with drift.

- (2) *Stable subordinator with drift.* Let  $\phi(\lambda) = b\lambda + \lambda^{\gamma/d_w}$ ,  $\gamma \in (0, d_w)$ ,  $b > 0$ . Then the corresponding subordinator is a sum of a pure drift subordinator  $bt$  and the pure jump  $\gamma/d_w$ -stable subordinator. In this case, **(L1)** and **(U2)** are satisfied with  $\alpha_1 = \alpha_2 = \beta = \delta = \gamma$ .
- (3) Let  $\phi(\lambda) = b\lambda + \lambda^{\gamma_1/d_w} [\log(1 + \lambda)]^{\gamma_2/d_w}$ ,  $\gamma_1 \in (0, d_w)$ ,  $\gamma_2 \in (-\gamma_1, d_w - \gamma_1)$ ,  $b > 0$ . In this case, we may take  $\alpha_1 = \beta = \gamma_1 + \gamma_2$ ,  $\alpha_2 = \gamma_1$  and  $\delta = (\gamma_1 + d_w)/2$  in **(L1)** and **(U2)**.

We now give examples of pure jump subordinators.

- (4) *Mixture of purely jump stable subordinators.* Let  $\phi(\lambda) = \sum_{i=1}^n \lambda^{\gamma_i/d_w}$ ,  $\gamma_i \in (0, d_w)$ ,  $n \in \mathbf{N}$ . One can directly check that **(L1)** and **(U3)** hold with  $\alpha_1 = \beta = \min_i \gamma_i$  and  $\alpha_2 = \delta = \max_i \gamma_i$ .
- (5) Let  $\phi(\lambda) = (\lambda + \lambda^{\gamma_1/d_w})^{\gamma_2/d_w}$ ,  $\gamma_1, \gamma_2 \in (0, d_w)$ . The conditions **(L1)** and **(U3)** hold with  $\alpha_1 = \beta = (\gamma_1 \gamma_2)/d_w$  and  $\alpha_2 = \delta = \gamma_2$ .
- (6) Let  $\phi(\lambda) = \lambda^{\gamma_1/d_w} [\log(1 + \lambda)]^{-\gamma_2/d_w}$ ,  $\gamma_1 \in (0, d_w)$ ,  $\gamma_2 \in (0, \gamma_1)$ . One can check that both assumptions **(L1)** and **(U3)** are fulfilled for  $\alpha_1 = \alpha_2 = \beta = \gamma_1 - \gamma_2$  and  $\delta = \gamma_1$ .

The last example satisfies our assumptions in part only. More precisely, it fulfils **(S1)**, **(S2)** and **(L1)**, but **(U1)**, **(U2)** nor **(U3)** do not hold.

- (7) *Relativistic  $\alpha/d_w$ -stable subordinator.* Let  $\phi(\lambda) = (\lambda + \vartheta^{d_w/\alpha})^{\alpha/d_w} - \vartheta$ ,  $\alpha \in (0, d_w)$ ,  $\vartheta > 0$ . The subordination via this subordinator leads to a very significant process called *relativistic  $\alpha$ -stable*. Here **(L1)** holds with  $\beta = d_w$ , but neither of the conditions **(U1)**, **(U2)** nor **(U3)** is satisfied. Theorem 3.1 can be applied to this process with such  $\beta$ , but our Theorem 4.1 does not cover this case. It can be conjectured that appropriate upper bounds hold true with the same rate  $\gamma = d_w$ . However, proving this would require more specialized arguments customized to the specific properties of the relativistic stable process.

We also provide examples of profile functions satisfying the assumptions of present paper.

**Example 6.2** [11, Example 4.1] Fix  $M_0 \in \mathbb{Z}$  and let the function  $\psi : \mathcal{G}_{M_0} \rightarrow [A, \infty)$  be such that  $\psi \in L^1(\mathcal{G}_{M_0}, m)$ , with  $A > 0$ . Define

$$W(x, y) := \begin{cases} \psi(\pi_{M_0}(y)), & \text{when } x, y \in \Delta_{M_0}(z_0), \text{ for some } z_0 \in \mathcal{G} \setminus \mathcal{V}_{M_0}, \\ 0, & \text{otherwise.} \end{cases}$$

It is established in [11] that **(W1)**–**(W3)** hold true. Clearly, **(W4)** holds as well.

**Example 6.3** [11, Example 4.2] Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be a function satisfying the following conditions.

- (1) There exists  $R > 0$  such that  $\varphi(x) = 0$  for all  $x \in (R, \infty)$ .
- (2) For every  $y \in \mathcal{G}$  one has  $\varphi(d(\cdot, y)) \in \mathcal{K}_{loc}^X$ .
- (3) There exist numbers  $a_0, A > 0$  such that  $\varphi(x) \geq A$  when  $x < a_0$ .

For such a function  $\varphi$  we define

$$W(x, y) := \varphi(d(x, y)), \quad x, y \in \mathcal{G}. \quad (6.1)$$

Again, **(W1)**–**(W3)** were verified in [11], and **(W4)** is straightforward.

We also give an example of profile functions  $W$  on  $\mathcal{G} \times \mathcal{G}$  with unbounded support satisfying our assumptions. Such profiles can be realized as follows.

**Example 6.4** First we set additional notation. Recall that for  $M \in \mathbb{Z}_+$  and every  $x \in (\mathcal{G} \setminus \mathcal{V}_M) \cup \{0\}$  there is exactly one triangle (the so-called natural cell) of size  $2^M$  in  $\mathcal{G}$ ,  $\Delta_M(x)$ , such that  $x \in \Delta_M(x)$ . If  $x \in \mathcal{V}_M \setminus \{0\}$ , then there are exactly two triangles  $\Delta_M^{(1)}(x)$  and  $\Delta_M^{(2)}(x)$  of size  $2^M$  such that  $\{x\} = \Delta_M^{(1)}(x) \cap \Delta_M^{(2)}(x)$ . For every  $x \in \mathcal{G}$  we define

$$r(x) := \sup_{\{p \in \mathcal{V}_0 : d(x, p) \leq 1\}} \sup \{M \in \mathbb{Z}_+ : p \in \mathcal{V}_M\}.$$

In particular,  $r(0) = \infty$ . One can check that for every  $x \in \mathcal{G}$  there is exactly one vertex  $p_x \in \mathcal{V}_0$  such that  $d(x, p_x) \leq 1$  and  $r(x) = \sup \{M \in \mathbb{Z}_+ : p_x \in \mathcal{V}_M\}$ . For  $x \in \mathcal{G}$  and  $M \in \mathbb{Z}_+$  we denote

$$D_M(x) = \begin{cases} \Delta_M^{(1)}(p_x) \cup \Delta_M^{(2)}(p_x) & \text{when } M \leq r(x) < \infty, \\ \Delta_M(x) & \text{when } M > r(x) \text{ or } p_x = 0. \end{cases}$$

Moreover, let  $(a_n)_{n \in \mathbb{Z}_+}$  be a nonincreasing sequence of nonnegative numbers such that

$$\sum_{M=1}^{\infty} \sum_{n=[M/4]+1}^{\infty} 2^{nd} a_n < \infty. \quad (6.2)$$

With the above notation we define

$$W(x, y) := \begin{cases} a_0 & \text{when } x \in \mathcal{G}, y \in D_0(x), \\ a_n & \text{when } x \in \mathcal{G}, y \in D_n(x) \setminus D_{n-1}(x), n = 1, 2, 3, \dots \end{cases}$$

Checking assumptions **(W1)** and **(W2)** for the profile  $W$  is an easy exercise. The geometric condition **(W3)** can be established by similar arguments as those in the justification in [11, Example 4.3]. To fulfil the condition **(W4)**, it is enough to assume that  $a_0, a_1 > 0$ . The decay conditions for the profile  $W$  as in (3.3) and (4.2) can be obtained by imposing some additional regularity on the elements of the sequence  $(a_n)_{n \in \mathbb{Z}_+}$  for large  $n$ . For instance, by taking  $a_n = 2^{-n(d+\theta)}$  with  $0 < \theta < \infty$ , we immediately get

$$2^{-d-\theta} = \liminf_{d(x,y) \rightarrow \infty} W(x, y) d(x, y)^{d+\theta} < \limsup_{d(x,y) \rightarrow \infty} W(x, y) d(x, y)^{d+\theta} = 1.$$

By putting  $a_n = 0$  for  $n \geq n_0$ , with some  $2 \leq n_0 \in \mathbb{Z}_+$ , we obtain a profile  $W$  with bounded support.

## 7 Appendix: the enlargement of obstacles method for Markov processes with compact state-space

The method of enlargement of obstacles was first introduced in [25] for diffusion processes on a compact state-space, evolving among killing obstacles. Its main ingredient is an estimate comparing the principal eigenvalue of the semigroup of such a process with the principal eigenvalue of the process evolving in a modified environment – with much bigger obstacles. It has been proven that under appropriate conditions on the process and on the configuration of the obstacle points, the principal eigenvalue does not increase significantly after such a modification, provided the principal eigenvalue of the initial process was not too big. The method was generalized to some non-diffusion Markov processes in [13]. We now need a version of these theorems for processes influenced by a killing potential with microscopic range, not microscopic killing obstacles.

### 7.1 The setting and the assumptions

Our initial setup consists of:

- \* a compact linear metric space  $(\mathcal{T}, d)$  equipped with a doubling Radon measure  $m$ , satisfying  $m(\mathcal{T}) = 1$ . More precisely, we assume that there exist  $r_0 > 0$  and  $C_d \geq 1$  such that for any  $x \in \mathcal{T}$  and  $0 < r < r_0$

$$m(B(x, r)) \leq C_d m(B(x, \frac{r}{3})), \quad (7.1)$$

- \* a right-continuous, strong Markov process  $X = (X_t, \mathbf{P}_x)_{t \geq 0, x \in \mathcal{T}}$  on  $\mathcal{T}$  with symmetric and strictly positive transition density  $p(t, x, y)$  with respect to  $m$  such that  $\forall t, \int_{\mathcal{T}} p(t, x, x) dm(x) < \infty$ ,
- \* a potential profile  $W : \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}_+$  of finite range: a measurable function with support included in  $\{(x, y) \in \mathcal{T} \times \mathcal{T} : d(x, y) \leq a\epsilon\}$ , where  $a > 0, \epsilon > 0$  are given, such that

$$\text{for every } t > 0 \text{ and } y \in \mathcal{T}, \quad \sup_{x \in \mathcal{T}} \mathbf{E}_x \int_0^t W(X_s, y) ds < \infty. \quad (7.2)$$

In applications,  $a$  will be considered fixed and  $\epsilon$  will tend to 0.

Suppose  $x_1, \dots, x_N \in \mathcal{T}$  are given points ('obstacles'), then one defines the potential  $V(x)$  as follows:

$$\mathcal{T} \ni x \mapsto V(x) = \sum_{i=1}^N W(x, x_i). \quad (7.3)$$

In applications, these points will be random and coming from a realisation of a Poisson point process  $\mathcal{N}$  on  $\mathcal{T}$ . Clearly, under the condition (7.2) we have  $\sup_{x \in \mathcal{T}} \mathbf{E}_x \int_0^t V(X_s) ds < \infty$ , for every  $t > 0$ .

Below we will study the process  $X$  perturbed by the potential  $V$ . Formally, we consider the Feynman-Kac semigroup  $(P_t^V)_{t \geq 0}$  on  $L^2(\mathcal{T}, m)$  consisting of symmetric operators

$$P_t^V f(x) = \mathbf{E}_x \left[ e^{-\int_0^t V(X_s) ds} f(X_t) \right], \quad f \in L^2(\mathcal{T}, m), \quad t > 0.$$

Operators  $P_t^V$  admit measurable, bounded and strictly positive kernels  $p^V(t, x, y)$ . Since also  $m(\mathcal{T}) = 1 < \infty$ , all  $P_t^V$  are of Hilbert-Schmidt type and have discrete spectra  $\{e^{-\lambda_k(V)t}\}_{k=1}^\infty$ , where  $0 \leq \lambda_1(V) < \lambda_2(V) < \dots \rightarrow \infty$  are eigenvalues of the generator of the semigroup  $(P_t^V)_{t \geq 0}$ . The corresponding eigenfunctions are denoted by  $\varphi_k^V$ . All  $\lambda_k(V)$  have finite multiplicity, the principal eigenvalue  $\lambda_1(V)$  is simple, and the ground state eigenfunction  $\varphi_1^V$  can be chosen to be strictly positive.

We intend to perform the following operation: for given  $b \gg a$  we would like to replace the support of the potential  $V$  by a much larger set  $\bigcup_{i=1}^N \overline{B}(x_i, b\epsilon)$ , and then to kill the initial process  $X$  when it enters this bigger set. Since  $\int_{\mathcal{T}} p(t, x, x) dx < \infty$ , the semigroup of this process again consists of symmetric Hilbert-Schmidt operators having discrete spectrum. We are interested in comparing the smallest eigenvalue of its generator with the principal eigenvalue  $\lambda_1(V)$  of the process  $X$  perturbed by the potential  $V$ . In general, we cannot enlarge every obstacle – we need to restrict our attention to those obstacles  $x_i$  that are well-surrounded by other obstacles (so-called *good obstacles*, see below). Other obstacles will be disregarded. Formally, we consider the sets

$$\mathcal{O}_b = \bigcup_{x_i \text{--good}} \overline{B}(x_i, b\epsilon), \quad \Theta_b = \mathcal{T} \setminus \mathcal{O}_b. \quad (7.4)$$

The process evolves now in the open set  $\Theta_b$  and is killed when it enters  $\mathcal{O}_b$ . Denote by  $\lambda_1(b)$  the smallest eigenvalue of the generator of this process.

The distinction between 'good' and 'bad' points is made as follows.

**Definition 7.1** Suppose  $b, \delta, R > 0$  are given, and let  $x_1, \dots, x_N$  be given obstacle points. Then  $x_{i_0}$  is called a *good obstacle point* if for all balls  $C = B(x_{i_0}, 10b\epsilon R^l)$  one has

$$m \left( \bigcup_{i=1}^N \overline{B}(x_i, b\epsilon) \cap C \right) \geq \frac{\delta}{C_d} m(C), \quad (7.5)$$

( $C_d$  is the constant from (7.1)) for all  $l = 0, 1, 2, \dots$ , as long as  $10b\epsilon R^l < r_0$ . Otherwise,  $x_{i_0}$  is called a *bad obstacle point*.



Formally speaking, this notion depends on  $b, \delta, R$ , but for the time being we do not incorporate these parameters into the notation.

Balls with centers at bad obstacle points sum up to a set with small volume.

**Lemma 7.1** [25, Lemma 1.3]

$$m\left(\bigcup_{x_i \text{--bad}} \overline{B}(x_i, b\epsilon)\right) \leq \delta. \quad (7.6)$$

We consider the following set of assumptions regarding the process  $X$  and the potential profile  $W$ .

**(P1)** There exists  $c_0 > 0$  such that  $\sup_{x, y \in \mathcal{T}} p(1, x, y) \leq c_0$ .

The remaining assumptions are concerned with recurrence properties of the process. We require that for any fixed  $a, b$ ,  $a \ll b$ ,  $b\epsilon < r_0$  and  $\delta > 0$  there exist constants  $\tau_0, c_1, c_2, c_3, \alpha, \kappa > 0$ ,  $R > 3$  and a nonincreasing function  $\phi : (0, r_0) \rightarrow (0, 1]$  such that:

**(P2)** for  $x, y \in \mathcal{T}$  with  $d(x, y) \leq b\epsilon$  one has

$$\mathbf{P}_x[\tau_{B(y, 10(R-2)b\epsilon)} < \frac{\tau_0 \epsilon^\alpha}{2}] < c_1;$$

**(P3)** when  $x, y \in \mathcal{T}$ , and  $d(x, y) \leq b\epsilon$ , then

$$\mathbf{E}_x[e^{-\int_0^{(\tau_0 \epsilon^\alpha)/2} W(X_s, y) ds}] \leq 1 - 2c_1;$$

**(P4)** for  $x, y \in \mathcal{T}$  satisfying  $d(x, y) \leq r\epsilon \leq r_0$  one has

$$\mathbf{P}_x[T_{B(y, b\epsilon)} \leq \frac{\tau_0 \epsilon^\alpha}{2}] \geq \phi(r);$$

**(P5)** for  $10b\epsilon \leq \beta \leq \frac{r_0}{R}$ , any points  $x, y \in \mathcal{T}$  with  $d(x, y) \leq \beta$ , and for any compact subset  $E \subset \mathcal{T}$  satisfying  $m(E \cap \overline{B}(y, \beta)) \geq \delta/C_d \cdot m(\overline{B}(y, \beta))$  one has

$$\mathbf{P}_x[T_E < \tau_{B(y, R\beta)}] \geq c_2;$$

**(P6)** for  $r < r_0/3$ ,  $A > 3r$  and  $x, y \in \mathcal{T}$  satisfying  $d(x, y) \leq r$  one has

$$\mathbf{P}_x[X_{\tau_{B(y, r)}} \notin B(y, A)] \leq c_3 \left(\frac{r}{A}\right)^\kappa.$$

Assumption **(P6)** was first introduced in [13]. It is typical for jump-type processes and was not needed in the diffusion case.

As a preparatory step, we relate the expression involving the term  $e^{-\int_0^t V(X_s) ds}$  to certain survival probability of the process. As in [26, page 171] (see also [20]), we attach exponential clocks to each of the points  $x_i$ , and then kill the process  $X_t$  once the quantity  $A_t^i := \int_0^t W(X_s, x_i) ds$  becomes bigger than this clock. More precisely: given configuration of points  $\{x_i\}_{i=1}^N$  we consider  $N$  Poisson processes  $(N^i(t))_{t \geq 0}$  with intensity 1 on the probability spaces  $(\Omega^i, \mathbf{P}^i)$  and the product measure  $\mathbf{P}_z^N := \mathbf{P}_z \otimes (\otimes_{i=1}^N \mathbf{P}^i)$  defined on the product of the canonical space for the process  $X$  and spaces  $\Omega^i$  of the processes  $(N^i(t))_{t \geq 0}$ , endowed with the product  $\sigma$ -algebra. Product measures  $\mathbf{P}_z^N$  turn the canonical process  $X$  and the canonical right continuous counting processes  $(N^i(t))_{t \geq 0}$  into independent processes, distributed respectively as the initial process  $X$  starting from  $z$  and Poisson counting

processes with unit intensity, starting from 0. They also satisfy the strong Markov property with respect to the appropriate filtration.

Let

$$T_i = \inf\{s \geq 0 : N^i(A_s^i) \geq 1\}, \quad \text{and} \quad T = \min_{i=1, \dots, N} T_i. \quad (7.7)$$

The following relation is central to our considerations:

$$\mathbf{P}_z^N[T > t | X] = \mathbf{P}_z^N[\forall i = 1, \dots, N \quad T_i > t | X] = \prod_i e^{-\int_0^t W(X_s, x_i) ds} = e^{-\int_0^t V(X_s) ds}. \quad (7.8)$$

Formally, in the next subsection we will be working with the measure  $\mathbf{P}_z^N$ . However, for simplicity, we do not indicate this in the notation, writing just  $\mathbf{P}_z$ .

## 7.2 The theorem comparing the bottoms of the spectra

The statement of the theorem together with its proof are very similar to that of [25, Theorem 1.4] and [13, Theorem 1] – the only difference is that we are concerned now with a killing potential, slowing down the process, and not killing obstacles of small radius. To make the paper self-contained, we state the theorem and briefly sketch the proof, indicating the changes that must be introduced and skipping the parts identical to those in previous papers.

Before we state the theorem, we make some additional technical preparation, needed in the càdlàg case. For given  $K > \delta > 0$  define

$$C(K, \delta) = e^K \left( 1 + c_0 \left( 1 + \frac{K}{\delta} \right) \right), \quad (7.9)$$

where  $c_0$  is the constant from the relation **(P1)**. Suppose that the number  $R$  entering assumptions **(P2)**, **(P5)** satisfies

$$\frac{c_3}{R^\kappa - 1} \leq \frac{1}{8} C(K, \delta)^{-1}. \quad (7.10)$$

This can be done without loss of generality: if **(P2)**, **(P5)** are satisfied with certain  $R > 0$ , then they are satisfied for any  $\tilde{R} > R$ .

**Theorem 7.1** *Assume that the process  $X_t$  is either:*

- (i) *a diffusion satisfying **(P1)** – **(P5)** or*
- (ii) *a discontinuous càdlàg process satisfying **(P1)** – **(P6)**, with  $R$  satisfying (7.10).*

*Let the numbers  $K > \delta > 0$ ,  $b \gg a$  be given. Then there exists  $\epsilon_0 = \epsilon_0(a, b, \delta, K, c_0, c_1, c_2, c_3, \alpha, \kappa)$  ( $c_3$  and  $\kappa$  not needed in the diffusion case) such that for any  $\epsilon < \epsilon_0$  ( $b\epsilon$  is the radius of obstacles in (7.4)) one has*

$$\lambda_1(b) \wedge K \leq \lambda_1(V) \wedge K + \delta. \quad (7.11)$$

**Proof.** We prove (i) and (ii) simultaneously.

Let  $m_0$  be the smallest possible integer for which

$$(1 - c_1 c_2)^{m_0} \leq \frac{1}{8} C(K, \delta)^{-1} \quad (\text{diffusion case (i)}) \quad (7.12)$$

or

$$(1 - c_1 c_2)^{\log_2 m_0} \leq \frac{1}{8} C(K, \delta)^{-1} \quad (\text{non-diffusion case (ii)}). \quad (7.13)$$

Then we set

$$D := 10bR^{m_0}. \quad (7.14)$$

Denote  $\lambda = \lambda_1(b) \wedge K - \delta$ . When  $\lambda \leq 0$ , there is nothing to prove, so we assume  $\lambda > 0$ . Our goal is to establish that

$$\int_{\mathcal{T}} \mathbf{E}_x[e^{\lambda T}] dm(x) < \infty, \quad (7.15)$$

where  $T$  is the stopping time introduced in (7.7).

The proof of (7.15) is divided in four steps.

STEP 1. For the stopping time

$$T_b = \inf\{t \geq 0 : X_s \in \mathcal{O}_b\}$$

we just repeat the argument from [25, pages 231-232] to get

$$\mathbf{E}_x[e^{\lambda T_b}] \leq e^K \left(1 + c_0 \left(1 + \frac{K}{\delta}\right)\right) = C(K, \delta), \quad (7.16)$$

for any  $x \in \mathcal{T}$ .

STEP 2. Denote  $\sigma_D = \inf\{t \geq 0 : X_t \notin \bigcup_{i=1}^N B(x_i, D\epsilon)\}$  and  $\tilde{T} = T \wedge \sigma_D$ . In this step we estimate  $\mathbf{E}_x[e^{\lambda \tilde{T}}]$ , for  $x \in \bigcup_{i=1}^N B(x_i, D\epsilon)$ .

Using the Fubini theorem we write

$$\mathbf{E}_x[e^{\lambda \tilde{T}}] = 1 + \int_0^\infty \lambda e^{\lambda u} \mathbf{P}_x[\tilde{T} > u] du, \quad x \in \bigcup_{i=1}^N B(x_i, D\epsilon). \quad (7.17)$$

The event  $\{\tilde{T} > u\}$  means that up to time  $u$  we have stayed inside  $\bigcup_{i=1}^N B(x_i, D\epsilon)$  and that  $T$  did not happen up to this moment. If  $T$  were to occur before  $u$ , we would have to enter the support of  $V$  before that time (an increase of some  $A_i^i$  can happen only when the process falls within the range of the potential  $V$ ). Pick  $x \in \bigcup_{i=1}^N B(x_i, D\epsilon)$  and let  $i_0$  be such that  $x \in B(x_{i_0}, D\epsilon)$ . Then (assumption **(P4)**)

$$\mathbf{P}_x[T_{B(x_{i_0}, b\epsilon)} \leq \frac{\tau_0 \epsilon^\alpha}{2}] \geq \phi(D)$$

and for  $y \in B(x_{i_0}, b\epsilon)$  (assumption **(P3)**)

$$\mathbf{P}_y[T \leq \frac{\tau_0 \epsilon^\alpha}{2}] = \mathbf{E}_y[1 - e^{-\int_0^{\tau_0 \epsilon^\alpha/2} V(X_s) ds}] \geq 1 - \mathbf{E}_y[e^{-\int_0^{\tau_0 \epsilon^\alpha/2} W(X_s, x_{i_0}) ds}] \geq 2c_1.$$

Then from the strong Markov property it follows that, once  $x \in \bigcup_{i=1}^N B(x_i, D\epsilon)$ ,

$$\mathbf{P}_x[\tilde{T} \leq \tau_0 \epsilon^\alpha] \geq 2c_1 \phi(D)$$

and from the ordinary Markov property we get

$$\mathbf{P}_x[\tilde{T} > u] \leq (1 - 2c_1 \phi(D))^{[u/(\tau_0 \epsilon^\alpha)]} \leq \frac{1}{1 - 2c_1 \phi(D)} (1 - 2c_1 \phi(D))^{u/(\tau_0 \epsilon^\alpha)}, \quad x \in \bigcup_{i=1}^N B(x_i, D\epsilon)$$

(when  $2c_1\phi(D) < 1$ ; when  $2c_1\phi(D) = 1$  then the quantity estimated is equal to 0). We now insert this estimate into (7.17) and proceed similarly as in [25, page 231], obtaining that for all  $x \in \bigcup_{i=1}^N B(x_i, D\epsilon)$ , as long as  $K\tau_0\epsilon^\alpha < \log(1 - 2c_1\phi(D))^{-1}$ ,

$$\mathbf{E}_x[e^{\lambda\tilde{T}}] \leq 1 + \frac{K\tau_0\epsilon^\alpha}{(1 - 2c_1\phi(D))[\log(1 - 2c_1\phi(D))^{-1} - K\tau_0\epsilon^\alpha]}$$

Now we set

$$\epsilon_0 = \frac{r_0}{4D} \wedge \inf \left\{ \epsilon > 0 : \frac{K\tau_0\epsilon^\alpha}{(1 - 2c_1\phi(D))[\log(1 - 2c_1\phi(D))^{-1} - K\tau_0\epsilon^\alpha]} > \frac{1}{8} C(K, \delta)^{-1} \right\},$$

so that when  $\epsilon \leq \epsilon_0$ , we have

$$\mathbf{E}_x[e^{\lambda\tilde{T}}] \leq 1 + \frac{1}{8} C(K, \delta)^{-1} \leq 2, \quad x \in \bigcup_{i=1}^N B(x_i, D\epsilon). \quad (7.18)$$

STEP 3. In this step, we finally estimate  $\mathbf{E}_x[e^{\lambda T}]$ , using estimates (7.16) and (7.18). Introduce the following sequence of stopping times:  $S_0 = 0$  and

$$S_1 = T_b + \tilde{T} \circ \theta_{T_b}, \quad S_{n+1} = S_n + S_1 \circ \theta_{S_n}, \quad n = 1, 2, \dots$$

Observe that each of the  $S_n$ 's (for  $n = 1, 2, \dots$ ) is realized at the moment  $t$  when either:  $X_t \in \text{supp } V$  (and the expression involving the potential gets bigger than the exponential clock) or  $X_t \notin \bigcup_{i=1}^N B(x_i, D\epsilon)$ . Since  $D > b > a$ , these two possibilities are distinct. Therefore, it makes sense to define

$$L = \inf\{n : X_{S_n} \in \text{supp } V\}$$

(with the convention  $\inf \emptyset = \infty$ ). By modifying the argument leading to (7.19) below combined with the Borel-Cantelli lemma, one can show that  $L$  is finite  $\mathbf{P}_x$ -a.s., for every  $x \in \mathcal{T}$ . We have  $T \leq S_L$ , and therefore  $\mathbf{E}_x[e^{\lambda T}] \leq \mathbf{E}_x[e^{\lambda S_L}]$ . We can write:

$$\begin{aligned} \mathbf{E}_x[e^{\lambda S_L}] &\leq 1 + \sum_{k=1}^{\infty} \mathbf{E}_x[e^{\lambda S_k} \mathbf{1}\{L = k\}] \\ &= 1 + \sum_{k=0}^{\infty} \mathbf{E}_x[e^{\lambda S_{k+1}} \mathbf{1}\{L = k+1\}] = (*). \end{aligned}$$

In view of the observation above,  $L = l$  means that  $X_{S_0}, \dots, X_{S_{l-1}} \in (\bigcup_{i \in I} B(x_i, D\epsilon))^c$  and  $X_{S_l} \in \text{supp } V$ . Therefore the estimate continues as

$$\begin{aligned} (*) &\leq 1 + \sum_{k=0}^{\infty} \mathbf{E}_x[\mathbf{E}_x[e^{\lambda(S_k + S_1 \circ \theta_{S_k})} \mathbf{1}\{X_{S_0}, \dots, X_{S_k} \notin \text{supp } V\} | \mathcal{F}_{S_k}]] \\ &= 1 + \sum_{k=0}^{\infty} \mathbf{E}_x[e^{\lambda S_k} \mathbf{1}\{X_{S_0}, \dots, X_{S_k} \notin \text{supp } V\} \cdot \mathbf{E}_{X_{S_k}}[e^{\lambda S_1}]]. \end{aligned}$$

For the last expectation, we first use the strong Markov property and then inequalities (7.16), (7.18):

$$\begin{aligned} \mathbf{E}_{X_{S_k}}[e^{\lambda S_1}] &= \mathbf{E}_{X_{S_k}}[e^{\lambda(T_b + \tilde{T} \circ \theta_{T_b})}] = \mathbf{E}_{X_{S_k}}[e^{\lambda T_b} \mathbf{E}_{X_{T_b}}[e^{\lambda \tilde{T}}]] \\ &\leq 2C(K, \delta). \end{aligned}$$

Consequently,

$$\mathbf{E}_x[e^{\lambda T}] \leq 1 + \sum_{k=0}^{\infty} \mathbf{E}_x[e^{\lambda S_k} \mathbf{1}\{X_{S_0} \notin \text{supp } V, \dots, X_{S_k} \notin \text{supp } V\}] \cdot 2C(K, \delta).$$

For  $k \geq 0$  we denote

$$a_k = \mathbf{E}_x[e^{\lambda S_k} \mathbf{1}\{X_{S_0} \notin \text{supp } V, \dots, X_{S_k} \notin \text{supp } V\}].$$

Our goal now is to show that for  $k = 1, 2, \dots$  we have  $a_k \leq \rho a_{k-1}$ , with certain constant  $\rho \in (0, 1)$ . This will do, as then we will have

$$\mathbf{E}_x[e^{\lambda T}] \leq 1 + \sum_{k=0}^{\infty} \rho^k \cdot 2C(K, \delta) = 1 + \frac{2C(K, \delta)}{1 - \rho} < \infty. \quad (7.19)$$

We can write, for  $k \geq 1$ ,

$$\begin{aligned} a_k &= \mathbf{E}_x[e^{\lambda S_k} \mathbf{1}\{X_{S_0} \notin \text{supp } V, \dots, X_{S_k} \notin \text{supp } V\}] \\ &= \mathbf{E}_x[e^{\lambda S_{k-1}} \mathbf{1}\{X_{S_0} \notin \text{supp } V, \dots, X_{S_{k-1}} \notin \text{supp } V\} \mathbf{E}_{X_{S_{k-1}}}[e^{\lambda S_1} \mathbf{1}\{S_1 \notin \text{supp } V\}]]. \end{aligned}$$

Therefore it suffices to find a universal bound on  $\mathbf{E}_z[e^{\lambda S_1} \mathbf{1}\{X_{S_1} \notin \text{supp } V\}]$ , for  $z \notin \text{supp } V$ .

Using again the strong Markov property, the inequality  $ab \leq (a - 1) + b$  (valid for  $a \geq 1$ ,  $b \leq 1$ ), (7.16), and (7.18), we write, for  $z \notin \text{supp } V$ :

$$\begin{aligned} \mathbb{E}_z[e^{\lambda S_1} \mathbf{1}\{S_1 \notin \text{supp } V\}] &\leq \mathbf{E}_z[e^{\lambda T_b} \mathbf{E}_{X_{T_b}}[e^{\lambda \tilde{T}} \mathbf{1}\{X_{\tilde{T}} \notin \text{supp } V\}]] \\ &\leq \mathbf{E}_z \left[ e^{\lambda T_b} \left( \mathbf{E}_{X_{T_b}}(e^{\lambda \tilde{T}} - 1) + \mathbf{P}_{X_{T_b}}(X_{\tilde{T}} \notin \text{supp } V) \right) \right] \\ &\leq \frac{1}{8} + \mathbf{E}_z \left[ e^{\lambda T_b} \mathbf{P}_{X_{T_b}}[X_{\tilde{T}} \notin \text{supp } V] \right] \\ &\leq \frac{1}{8} + C(K, \delta) \sup_{x \in \mathcal{O}_b} \mathbf{P}_x[X_{\tilde{T}} \notin \text{supp } V]. \end{aligned} \quad (7.20)$$

When  $x \in \mathcal{O}_b$ , then  $x$  lies at a distance at most  $b\epsilon$  from a good point, say  $x_{j_0}$ . We have:

$$\mathbf{P}_x[X_{\tilde{T}} \notin \text{supp } V] = \mathbf{P}_x[\tilde{T} = \sigma_D] = \mathbf{P}_x[T > \sigma_D] \leq \mathbf{P}_x[T > \tau_{B(x_{j_0}, D\epsilon)}]. \quad (7.21)$$

Identically as in [25, page 233] and [13, pages 742-743] we obtain that

$$\begin{aligned} \forall k > 0 \quad \forall x \in \mathcal{T} \text{ such that } d(x, x_{j_0}) \leq 10b\epsilon R^l \\ \text{one has } \mathbf{P}_x[T < \tau_{B(x_{j_0}, 10b\epsilon R^{l+1+k})}] \geq c_1 c_2 > 0. \end{aligned} \quad (7.22)$$

In the diffusive case, applying the strong Markov property at moments  $\tau_{B(x_{j_0}, 10b\epsilon R^l)}$ ,  $l = 1, 2, \dots, m_0 - 1$  we get that for  $x \in B(x_{j_0}, b\epsilon)$  one has

$$\mathbf{P}_x[T > \tau_{B(x_{j_0}, D\epsilon)}] \leq (1 - c_1 c_2)^{m_0} \leq \frac{1}{8} C(K, \delta)^{-1}.$$

In the non-diffusive case, we use estimates from [13, pages 743-745] and get

$$\mathbf{P}_x[T > \tau_{B(x_{j_0}, D\epsilon)}] \leq (1 - c_1 c_2)^{\log_2 m_0} + \frac{c_3}{R^\kappa - 1} \leq \frac{1}{4} C(K, \delta)^{-1}.$$

In either case we have

$$\sup_{x \in \mathcal{O}_b} \mathbf{P}_x[X_{\tilde{T}} \notin \text{supp } V] \leq \frac{1}{4} C(K, \delta)^{-1},$$

which inserted into (7.20) results in the estimate

$$\sup_{z \notin \text{supp } V} \mathbf{E}_z[e^{\lambda S_1} \mathbf{1}\{S_1 \notin \text{supp } V\}] \leq \frac{3}{8} (=:\rho).$$

Relation (7.19) follows.

STEP 4. THE CONCLUSION. As the estimate obtained in Step 3 is uniform in  $x \in \mathcal{T}$ , inequality (7.15) follows as well. From the Fubini theorem we have

$$\int_{\mathcal{T}} \mathbf{E}_x[e^{\lambda T}] dm(x) = 1 + \int_0^\infty \lambda e^{\lambda v} \int_{\mathcal{T}} \mathbf{P}_x[T > v] dm(x) dv.$$

Hence,

$$\begin{aligned} \infty > \int_0^\infty \lambda e^{\lambda v} \int_{\mathcal{T}} \mathbf{P}_x[T > v] dm(x) dv &\geq \int_0^\infty \lambda e^{\lambda v} \sum_k \langle \varphi_k^V, \mathbf{1} \rangle_{L^2(\mathcal{T}, m)}^2 e^{-\lambda_k^V v} dv \\ &\geq \langle \varphi_1^V, \mathbf{1} \rangle_{L^2(\mathcal{T}, m)}^2 \lambda \int_0^\infty e^{(\lambda - \lambda_1^V)v} dv. \end{aligned}$$

Since  $\langle \varphi_1^V, \mathbf{1} \rangle_{L^2(\mathcal{T}, m)} > 0$ , the last integral is finite, and so  $\lambda < \lambda_1^V$ . The proof is concluded.  $\square$

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